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LETTER

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Relating quantum mechanics with hydrodynamic turbulence

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Abstract – In this letter we attempt to trace back the origin of quantum uncertainty. We show that the Schrödinger equation can be mapped into the inviscid Favre-Reynolds turbulence equations of classical compressible fluids, albeit in zero temperature. Under this mapping the probability density function becomes the Reynolds time mean density of the fluid, the real and the imaginary parts of the momentum become the mean and turbulent root-mean-square velocities, respectively, where the latter obeys the first Fick law of diffusion and saturates the lower bound of the uncertainty principle. The mean pressure is proportional to the divergence of the turbulent mass flux and is the source for stochasticity. The roles of the pressure gradient force and the Reynolds stress tensor convergence, under this mapping, are illustrated in two well-known systems, namely, the $1s$ orbital hydrogen atom and the 1D dynamic Gaussian wavepacket. Finally, we analyze within an independent part of the letter, a conjecture according to which this pressure results from vacuum fluctuations at the zero-point energy, mediated by random collisions of the particle with virtual photons. This suggests that the typical turbulent eddy is of the size of the Compton wavelength corresponding to a Reynolds averaging time scale which is twice the Zitterbewegung period. Moreover, according to this interpretation the quantized characteristics of the particle result from interactions with virtual photons.

Introduction. – Quantum mechanics appears to be an inherently uncertain theory, significantly different from classical mechanics. However, the probabilistic nature of quantum mechanics may suggest that it does share similarities with mean theories such as thermodynamics and hydrodynamics. The various analogies between quantum mechanics and hydrodynamics, first introduced by Madelung [1], and later further developed, *e.g.*, by [2–4], is still under active research [5–7]. This hydrodynamic viewpoint is tightly related to Bohmian mechanics [8–10] and stochastic quantum mechanics [11–23].

The term "quantum turbulence" is usually related to the motion of quantum fluids such as superfluid helium and atomic Bose-Einstein condensates, in the presence of quantized vortices, and a two-fluid dynamics at finite temperature [24]. We show below that the non-relativistic Schrödinger equation (SE) of a single particle in a vacuum can be mapped by itself into the turbulent equations of a classical fluid in the limit of zero temperature and zero viscosity. We then illustrate the consequences of this mapping using two well-known systems. We conclude with a brief interpretation of these results and a discussion.

Mapping of the Schrödinger equation to turbulence equations. – The SE, in the presence of an external scalar potential U,

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\frac{P^2}{2m} + U\right)\Psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + U\right)\Psi, \qquad (1)$$

where

$$\Psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{iS(\mathbf{x},t)/\hbar}, \qquad (2)$$

can be written in its Madelung [1] form (here and below summation over repeated indices is implied):

$$\frac{\partial\rho}{\partial t} = -\frac{\partial\left(\rho v_j\right)}{\partial x_j}, \qquad (3a)$$

$$m\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}\right)v_i = -\frac{\partial}{\partial x_i}\left(U + Q\right), \qquad (3b)$$

where

$$v_i \equiv \Re\left(\psi^{-1}P\psi\right) = \Re\left(-i\hbar\frac{\partial ln\Psi}{\partial x_i}\right) = \frac{1}{m}\frac{\partial S}{\partial x_i}, \quad (4)$$

and

$$Q = -\frac{\hbar^2}{2m\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x_j^2} \quad (5)$$

is the Bohm potential [8,25]. Using now the following identity for a scalar field α:

$$\frac{1}{\alpha}\frac{\partial^2\alpha}{\partial x_j^2} = \frac{\partial^2 \ln\alpha}{\partial x_j^2} + \left(\frac{\partial \ln\alpha}{\partial x_j}\right)^2, \quad (6)$$

and defining Fick's diffusion velocity with the Nelson diffusivity coefficient $\left(\frac{\hbar}{2m}\right)$ [13]:

$$w_i \equiv \Im\left(\psi^{-1}P\psi\right) = -\left(\frac{\hbar}{2m}\right)\frac{\partial ln\rho}{\partial x_i}, \quad (7)$$

eq. (3b) can be written after some vector calculus as

$$\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}\right)v_i = -\frac{1}{m\rho}\frac{\partial}{\partial x_j}\left(p\delta_{ij} + m\rho w_i w_j\right) - \frac{1}{m}\frac{\partial U}{\partial x_i}, \quad (8)$$

where

$$p = \left(\frac{\hbar}{2m}\right)\frac{\partial}{\partial x_j}\left(m\rho w_j\right) = -\left(\frac{\hbar}{2m}\right)^2\frac{\partial^2\left(m\rho\right)}{\partial x_j^2}. \quad (9)$$

Consider now the continuity and momentum equations for classical Eulerian fluids:

$$\frac{\partial\rho_m}{\partial t} = -\frac{\partial}{\partial x_j}\left(\rho_m u_j\right), \quad (10a)$$

$$\left(\frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}\right)u_i = -\frac{1}{\rho_m}\frac{\partial\pi}{\partial x_i} - \frac{1}{m}\frac{\partial U}{\partial x_i}, \quad (10b)$$

where $\mathbf{u}$ is the hydrodynamic velocity, π is the pressure and ρ_m denotes the fluid mass density (to distinguish it from the quantum probability density function ρ). If the fluid is turbulent we can decompose the mean dynamics from the turbulent one by imposing on (10) both the Reynolds time averaging on a typical turbulent eddy time scale τ:

$$\overline{f} \equiv \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f\mathrm{d}t, \quad (11a)$$

and the Favre density weighted time averaging (suitable for compressible flow [26]):

$$\tilde{f} \equiv \frac{\overline{\rho_m f}}{\overline{\rho}_m} \quad (11b)$$

(so that $f = \overline{f} + f' = \tilde{f} + f''$, where $\overline{f'} = \widetilde{f''} = 0$). This provides the compressible turbulent equations (without closure) for the high-Reynolds-number regime:

$$\frac{\partial\overline{\rho}_m}{\partial t} = -\frac{\partial}{\partial x_j}(\overline{\rho}_m\tilde{u}_j), \quad (12a)$$

$$\left(\frac{\partial}{\partial t} + \tilde{u}_j\frac{\partial}{\partial x_j}\right)\tilde{u}_i = -\frac{1}{\overline{\rho}_m}\frac{\partial}{\partial x_j}\left(\overline{\pi}\delta_{ij} + \overline{\rho}_m\widetilde{u_i''u_j''}\right) - \frac{1}{m}\frac{\partial U}{\partial x_i}, \quad (12b)$$

where the second term in the RHS is the turbulence Reynolds stress tensor convergence. Toward a closure (*i.e.*, an expression of the turbulent fluxes in terms of the mean flow fields) we define the turbulent root-mean-square velocity (RMS) $\hat{\mathbf{u}}$ to satisfy

$$\hat{u}_i\hat{u}_j \equiv \widetilde{u_i''u_j''}, \quad (13)$$

which allows to write (10b) as

$$\left(\frac{\partial}{\partial t} + \tilde{u}_j\frac{\partial}{\partial x_j}\right)\tilde{u}_i = -\frac{1}{\overline{\rho}_m}\frac{\partial}{\partial x_j}\left(\overline{\pi}\delta_{ij} + \overline{\rho}_m\hat{u}_i\hat{u}_j\right) - \frac{1}{m}\frac{\partial U}{\partial x_i}. \quad (14)$$

Comparing (3a) with (12a) and (8) with (14) implies that for the Madelung fluid

$$\overline{\rho}_m = m\rho; \qquad \tilde{\mathbf{u}} = \mathbf{v}; \qquad \hat{\mathbf{u}} = \mathbf{w}; \qquad \overline{\pi} = p. \quad (15)$$

Examples. – We consider two well-known simple examples in order to obtain some better understanding and further insights regarding the derivations above (for other examples see [27]).

The first is the $1s$ orbital wave function of the hydrogen atom:

$$\Psi = \frac{e^{-(r/a_0)}}{\sqrt{\pi a_0^3}}e^{-iEt/\hbar} \quad (16)$$

(where a_0 is the Bohr radius, $E = -\hbar^2/2m_e a_0^2$ is the eigenstate energy and m_e is the electron mass). Since S is not a function of space $\tilde{\mathbf{u}} = 0$, the electron kinetic energy is only the TKE one. The magnitude of the turbulent velocity possesses only a radial component $\hat{u}_r = \hbar/m_e a_0$, thus the electron TKE, $\hat{K} = -E$. The electric potential $U = -\hbar^2/m_e a_0 r$, so that $Q = \hat{K} + p/\rho = E - U$. As can be verified from (9) indeed $m\overline{\pi}/\overline{\rho}_m = 2E - U$, so that $\langle Q\rangle = \langle\hat{K}\rangle$ and $\langle U\rangle = \langle 2E\rangle$. It is interesting to understand the force balance in (12b), for this mean equilibrium state, among the pressure gradient force (PGF), the Reynolds stress convergence term (written in spherical coordinates) and the Coulomb force:

$$0 = -\frac{1}{\overline{\rho}_m}\frac{\partial\overline{\pi}}{\partial r} - \left(\frac{2}{r} + \frac{\partial\ln\overline{\rho}_m}{\partial r}\right)\hat{u}_r^2 - \frac{1}{m_e}\frac{\partial U}{\partial r}. \quad (17)$$

The mean pressure

$$\overline{\pi} = \frac{\hbar^2}{m_e^2 a_0}\left(\frac{1}{r} - \frac{1}{a_0}\right)\overline{\rho}_m, \quad (18)$$

is positive (negative) inside (outside) the Bohr radius. Nevertheless the PGF is not always pointed outward to balance the inward Coulomb force. In fact the Reynolds stress convergence term becomes

$$2\left(\frac{\hbar}{m_e a_0}\right)^2\left(\frac{1}{a_0} - \frac{1}{r}\right) = \frac{1}{\overline{\rho}_m}\frac{\partial\overline{\pi}}{\partial r} + \frac{1}{m_e}\frac{\partial U}{\partial r}, \quad (19)$$

which is negative (positive) inside (outside) the Bohr radius. Thus, inside (outside) the Bohr radius the PGF is too strong (weak) to balance the Coulomb force. This imbalance, everywhere but the Bohr radius itself, is compensated by the Reynolds stress convergence.

Hence, in terms of this formulation the electron exhibits a random radial motion which draws a picture equivalent to the spherical plot of the probability density function. This stands in contrast both with the Bohr model, where the electron rotates around the nucleus on a plane, and with the Bohmian mechanics interpreting the electron at the s orbitals to stay fixed at the Bohr radius (as $\dot{\mathbf{X}} \equiv \tilde{\mathbf{u}} = 0$).

The second example is the 1D dynamic Gaussian solution to (1) for a free particle:

$$\begin{aligned}\rho(x,t) &= \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma}; \qquad S = \frac{mx^2}{2}\frac{\partial \ln\sigma}{\partial t};\\ \sigma^2 &= \sigma_0^2 + \left(\frac{\hbar t}{2m\sigma_0}\right)^2.\end{aligned} \tag{20}$$

The mean velocity

$$\tilde{u} = \frac{xt}{\left(\frac{m\sigma_0^2}{\hbar}\right) + t^2} \tag{21}$$

diverges from the Gaussian center, although the Gaussian is only spreading in time without changing its mean position. In fact this divergence yields the decrease in density according to the continuity equation (3a). The turbulent velocity,

$$\hat{u} = \left(\frac{2m\sigma_0^2}{\hbar t}\right)\tilde{u}, \tag{22}$$

indicates that the turbulence intensity $|\tilde{u}/\hat{u}|$ decreases uniformly in time as $1/t$. The material acceleration of the mean velocity,

$$\frac{D\tilde{u}}{Dt} = \left(\frac{\partial}{\partial t} + \tilde{u}\frac{\partial}{\partial x}\right)\tilde{u} = -\frac{1}{m}\frac{\partial Q}{\partial x} = \left(\frac{\hbar}{2m}\right)^2\frac{x}{\sigma^4}, \tag{23}$$

diverges linearly as well from the mean position. Bearing an interesting similarity to the previous example, the mean pressure,

$$\bar{\pi} = \left(\frac{\hbar}{2m\sigma}\right)^2\left[1 - \left(\frac{x}{\sigma}\right)\right]\bar{\rho}_m, \tag{24}$$

is positive inside when $|x| < \sigma$ and negative outside of it. The mean momentum equation (12b) for this case is therefore

$$\frac{D\tilde{u}}{Dt} = -\frac{1}{\bar{\rho}_m}\frac{\partial\bar{\pi}}{\partial x} - \frac{1}{\bar{\rho}_m}\frac{\partial}{\partial x}\left(\bar{\rho}_m\hat{u}^2\right), \tag{25}$$

where

$$-\frac{1}{\bar{\rho}_m}\frac{\partial\bar{\pi}}{\partial x} = \frac{D\tilde{u}}{Dt}\left[3 - \left(\frac{x}{\sigma}\right)^2\right] \tag{26}$$

and

$$-\frac{1}{\bar{\rho}_m}\frac{\partial}{\partial x}\left(\bar{\rho}_m\hat{u}^2\right) = \frac{D\tilde{u}}{Dt}\left[\left(\frac{x}{\sigma}\right)^2 - 2\right]. \tag{27}$$

Similarly to the previous example, the Reynolds stress convergence hinders the mean PGF near the center and helps it at the flanks of the density distribution.

Interpretation. The mapping above suggests that the SE may be interpreted as a description of the mean dynamics of a turbulent inviscid compressible flow. Under this interpretation the quantum probability density function is the time mean density (per unit mass) of the fluid, the real part of the momentum per unit mass (proportional to the wave function phase gradient) is the Favre-Reynolds mean velocity, where the imaginary part provides the turbulent RMS velocity. The mean pressure is proportional to the divergence of the turbulent mass flux, so that the turbulent flux diverges (converges) from (into) positive (negative) mean pressure anomalies.

Furthermore, the conserved energy resulting from the SE can be written as [28]

$$\begin{aligned}\langle E\rangle &= \int \Psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + U\right)\Psi \mathrm{d}^3x = \\ &\int \rho\left(\tilde{K} + Q + U\right)\mathrm{d}^3x,\end{aligned} \tag{28}$$

where

$$\tilde{K} \equiv \frac{m\tilde{u}_i^2}{2}, \tag{29}$$

is commonly considered as the mean kinetic energy (MKE). When similarly defining the turbulent kinetic energy (TKE) as

$$\hat{K} \equiv \frac{m\hat{u}_i^2}{2}, \tag{30}$$

it can be shown, after some algebra, that

$$Q = \left(\hat{K} + m\frac{\overline{\pi}}{\overline{\rho}_m}\right). \tag{31}$$

Thus, the Bohm potential can be interpreted as the enthalpy of a turbulent fluid with zero temperature. Since the domain integrated work performed by the pressure fluctuation p vanishes $\langle Q\rangle = \langle\hat{K}\rangle$, the energy expectation value of the quantum particle becomes

$$\langle E\rangle = \int \rho\left(\tilde{K} + \hat{K} + U\right)\mathrm{d}^3x, \tag{32}$$

which is the total energy of a turbulent fluid in zero temperature.

The partition between the turbulent and the mean parts of the flow suggests that the SE can provide information only on the mean dynamics. The turbulent RMS velocity $\hat{\mathbf{u}}$ correlation with the position saturates to the lower bound of the uncertainty principle as $\langle m\hat{u}_i x_i\rangle = 3\hbar/2$. The pressure in the fluid-like description is the source for stochasticity. Taking $\pi = 0$ in (10) we obtain the Newtonian dynamics solution: $m\ddot{X}_j = -\frac{\partial U}{\partial x_j}$, $\rho_m = m\delta(\mathbf{x} - \mathbf{X}(t))$, where $\mathbf{X}(t)$ is the particle location and $\mathbf{u} = \dot{\mathbf{X}}$. Thus, a plausible interpretation of this analysis is that the SE represents the mean motion of a particle which results from a stochastic forcing exerted by fluctuations whose mean pressure is determined by (15).

We shall now examine an hypothesis according to which the source of this pressure is the vacuum fluctuations of the zero-point energy. A small and light enough particle in a vacuum might be expected to be affected by random collisions with virtual particles [21]. In this respect, it is interesting to determine the speed of sound in the turbulent fluid. The Fourier transform of the pressure from eq. (15) leads to

$$\overline{\pi}_k = \overline{\rho}_{m,k} \left(\frac{\hbar k}{2m} \right)^2, \tag{33}$$

which defines the speed of sound as

$$c_k \equiv \left(\frac{\partial \overline{\pi}_k}{\partial \overline{\rho}_{m,k}} \right)^{1/2}_{\mathcal{S}} = \frac{\hbar k}{2m}, \tag{34}$$

where the derivative is taken at constant entropy $\mathcal{S}$. Hence, the speed of sounds depends on the wavelength, in contrast to classical fluids. The corresponding momentum is equal to the momentum of the incident virtual particles due to the conservation law. Since they should be relativistic virtual particles moving with the speed of light c, their energy is equal to $E = c\hbar k/2 = \hbar\omega/2$. Therefore, the stochastic pressure π could result from the zero-point energy of the virtual particles in vacuum. We note that c_k is the phase velocity, while the group velocity is $2c_k = \hbar k/m$, *i.e.*, the Fourier image of the quantum operator divided by the mass P/m.

Next we speculate what should be the typical turbulent time averaging τ. Since the turbulent RMS velocity (15) is a diffusive velocity with a diffusion coefficient of $D = \hbar/2m$ we may assume that $\tau = l^2/2D = ml^2/\hbar$, where l is the typical size of the turbulent eddy. Since the particle chaotic motion results from random interactions with virtual photons, we assume further that the eddy turnover time scale is $\tau = l/c$, yielding a typical eddy size of the reduced Compton wavelength $l = \hbar/mc$, where $\tau = \hbar/mc^2$ corresponds to the angular frequency $\Omega = mc^2/\hbar$, which is half of the Zitterbewegung frequency (see also [22]).

Discussion. – In this letter we mapped the nonrelativistic Schrödinger equation of a single particle in a vacuum into the turbulent equations of a compressible inviscid fluid at zero temperature. The aim was not just an intellectual exercise but an attempt to understand the reason for the probabilistic nature of the SE as well as the source for quantization. If we accept the paradigm of a fluctuating vacuum at the zero-point temperature, then we may assume that the virtual photons occupying the vacuum randomly hit the particle and impart their energy and momentum on the particle (the discussed particles are assumed to have no internal degrees of freedom hence all collisions were assumed to be elastic). The random nature of these collisions is the source for uncertainty and since the photons' energy and momentum are quantized this reflects on the particle's stochastic behavior. The reason for the fluid-like representation of the SE in the Madelung equation is that the overall effect of the collisions of the photons on the particle exert pressure, whereas the reason for the turbulence-like form equations is the existence of a short time scale, proportional to the inverse of the Zitterbewegung frequency, that separates between the random motion of the particle due to interaction with the photons and the particle mean motion. According to this interpretation, in multipartite scenarios (described within multidimensional phase space), quantum nonlocality would emerge similarly to the de Broglie-Bohm interpretation due to the quantum potential. However, our approach may enable to track nonlocality down to spatial correlations within the vacuum.

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REFERENCES

[1] Madelung E., *Z. Phys. A*, **40** (1927) 322.
[2] Takabayasi T., *Prog. Theor. Phys.*, **143** (1952) 8.
[3] Schönberg M., *Nuovo Cimento*, **12** (1954) 103.
[4] Sonego S., *Found. Phys.*, **21** (1991) 1135.
[5] Couder Y., Protiere S., Fort E. and Boudaoud A., *Nature*, **437** (2005) 208.
[6] Sáenz J. P., Cristea-Platon T. and Bush J. W. M, *Nat. Phys.*, **14** (2018) 315.
[7] Bohr T., *Nat. Phys.*, **14** (2018) 209.
[8] Bohm D., *Phys. Rev.*, **85** (1952) 166.
[9] Bohm D. and Hiley B. J., *The Undivided Universe: an Ontological Interpretation of Quantum Theory* (Routledge, London) 1993.
[10] Holland P. R., *The Quantum Theory of Motion: An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics* (Cambridge University Press, Cambridge) 1993.
[11] Fényes I., *Acta Bolyaina*, **1** (1946) 5.
[12] Fényes I., *Z. Phys.*, **132** (1952) 81.
[13] Nelson E., *Phys. Rev.*, **150** (1966) 1079.
[14] de la Peña L., *Phys. Lett. A*, **24** (1967) 603.
[15] de la Peña L., *J. Math. Phys.*, **10** (1969) 1620.
[16] Davidson M. P., *Lett. Math. Phys.*, **3** (1979) 271.
[17] Davidson M. P., *J. Math. Phys.*, **20** (1979) 1865.
[18] Guerra F., *Phys. Rep.*, **77** (1981) 263.
[19] Guerra F. and Morato L. M., *Phys. Rev. D*, **27** (1983) 1774.
[20] de la Peña L., Cetto A. M. and Valdes-Hernandez A., *The Emerging Quantum: The Physics behind Quantum Mechanics* (Springer, New York) 2015.
[21] Tsekov R., *J. Phys.: Conf. Ser.*, **701** (2016) 012034.

[22] Tsekov R., *Fluct. Noise Lett.*, **15** (2016) 1650022.
[23] Tsekov R., Heifetz E. and Cohen E., *Fluct. Noise Lett.*, **16** (2017) 1750028.
[24] Barenghi C. F., Skrbek L. and Sreenivasan K. R., *Proc. Natl. Acad. Sci. U.S.A.*, **111** (2014) 4647.
[25] Dennis G., de Gosson M. and Hiley B., *Phys. Lett. A*, **379** (2015) 1224.
[26] Lele S. K., *Annu. Rev. Fluid Mech.*, **26** (1994) 211.
[27] Wilhelm H. E., *Phys. Rev. D*, **1** (1970) 2278.
[28] Heifetz E. and Cohen E., *Found. Phys.*, **45** (2015) 1514.