

## On the mechanism of self gravitating Rossby interfacial waves in proto-stellar accretion discs

Ron Yellin-Bergovoy <sup>†1</sup>, Eyal Heifetz<sup>1</sup> & Orkan M. Umurhan<sup>2</sup>

<sup>1</sup>Department of Earth Sciences, Tel-Aviv University, Tel-Aviv, Israel

<sup>2</sup>NASA Ames Research Center, Division of Space Sciences, Planetary Systems Branch, Moffett Field, CA, 94035 USA

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The dynamical response of edge waves under the influence of self-gravity is examined in an idealized two-dimensional model of a proto-stellar disc, characterized in steady state as a rotating vertically infinite cylinder of fluid with constant density except for a single density interface at some radius  $r_0$ . The fluid in basic state is prescribed to rotate with a Keplerian profile  $\Omega_k(r) \sim r^{-3/2}$  modified by some additional azimuthal sheared flow. A linear analysis shows that there are two azimuthally propagating edge waves, kin to the familiar Rossby waves and surface gravity waves in terrestrial studies, which move opposite to one another with respect to the local basic state rotation rate at the interface. Instability only occurs if the radial pressure gradient is opposite to that of the density jump (unstably stratified) where self-gravity acts as a wave stabilizer irrespective of the stratification of the system. The propagation properties of the waves are discussed in detail in the language of vorticity edge waves. The roles of both Boussinesq and non-Boussinesq effects upon the stability and propagation of these waves with and without the inclusion of self-gravity are then quantified. The dynamics involved with self-gravity non-Boussinesq effect is shown to be a source of vorticity production where there is a jump in the basic state density, in addition, self-gravity also alters the dynamics via the radial main pressure gradient, which is a Boussinesq effect. Further applications of these mechanical insights are presented in the conclusion including the ways in which multiple density jumps or gaps may or may not be stable.

**Keywords:** protoplanetary accretion disc, Rossby wave instability, self gravity, vorticity waves

### 1. Introduction

The Atacama Large Millimeter/submillimeter Array (ALMA) has opened up a new window revealing the structure of proto-planet accretion discs (pp-discs hereafter). Most notable of the recently reported ALMA discoveries is the appearance of asymmetric “peanut” features in the outer ( $> 80$  AU) regions of several pp-disc systems including SAO 206462, SR 21, LKH $\alpha$  330 OPH IRS 48 and HD 142527 (Casassus *et al.* 2013, Fukagawa *et al.* 2013, Isella *et al.* 2013, van der Marel *et al.* 2013, Pérez *et al.* 2014). The ALMA website recently published a high resolution image of HL Tauri, a debris-disc phase T-Tauri system, showing a disc with alternating axisymmetric rings and gaps with unambiguous non-axisymmetric features set atop. While planets have yet to be detected in the HL Tauri system, the gap-ring features appear to indicate an ongoing process similar to the dynamics responsible for the Kirkwood gaps in the Jupiter-Asteroid-belt system, in which an embedded (as yet unseen) planet drives gap creation through resonant gravitational forcing of an otherwise structurally uniform Keplerian disc (A. Parker, private communication). Furthermore, the furthest parts of these same discs are likely to be sufficiently cold enough that the densities reached in their local disc midplane’s are correspondingly high enough such that self-gravitational effects should not be ignored. The dynamical stability of sheared dynamical structures like those seen in HL Tauri is the main inspiration for this work: we seek to develop a theoretical framework that could be helpful in interpreting the dynamical

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<sup>†</sup>Corresponding author. Email: yellinr@post.tau.ac.il

nature of structures emerging in mainly cold pp-disc systems in which self-gravity of the disc gas plays an important dynamical role.

Some pp-disc systems exhibiting peanut features also show indications of severe dust depletion within their inner radial locations. It has been suggested that pressure/density enhancements in the outer parts of the disc act as particle traps preventing the inner parts of these discs from having their dust replenished by accretion from the outer parts. Interestingly, axisymmetric pressure enhancements, provided they meet minimum requirements upon their amplitudes and radial girth, are known to undergo a barotropic shear transition known as the Rossby Wave Instability (RWI hereafter, (Lovelace *et al.* 1999, Li *et al.* 2000, 2001)). This instability leads to the creation of coherent anticyclonic vortex structures with wide azimuthal extent. From both theoretical and numerical/computational considerations it is also understood that anticyclonic vortex structures in pp-discs readily entrain dust through well-known drag-momentum exchange mechanisms (Barge and Sommeria 1995). While the origins of the density enhancement and the causal dilemma posed by the above scenario remains unsettled (see recent discussion in Flock *et al.* (2014)), the fact remains that rings/gaps and (generally speaking) *nearly axisymmetric annular structures* are the basic states upon which dynamical evolution proceeds in pp-discs. A detailed examination of the dynamical state of such structures from a physically mechanistic point of view is therefore of interest to theoreticians and observers alike. While the dynamical stability of self-gravitating disc systems is a vast subject that we cannot possibly cover here (see the reviews of Armitage (2011)), we are interested in examining the mechanistic nature of stability in self-gravitating zonal flows in accretion discs. Steady axisymmetric disc zonal flows arise from the balance between the Coriolis and centrifugal effects and the radial dependencies in the pressure and self gravity. Indeed, inspection of the HL Tauri system suggests that the flow within the rings and gaps should exhibit pronounced zonal flow structure since the distribution of matter appears nearly axisymmetric, yet radially non-uniform.

Barotropic zonal flows can become unstable and the RWI is one example. Barotropic shear instabilities, i.e., instabilities in the absence of baroclinic torque but in the presence of significance rotation may be interpreted within the framework of potential vorticity dynamics and the action of counter propagating Rossby waves (Bretherton 1966, Hoskins *et al.* 1985, Baines and Mitsudera 1994, Heifetz *et al.* 1999, to name just a few), and the RWI has been shown to be consistently interpreted within that framework (Umurhan 2010). Potential vorticity<sup>1</sup> (PV) in this context is equal to  $q/\rho$  in which  $q \equiv (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}}$  is the vorticity in the disc vertical direction and  $\rho$  is the density. The instantaneous basic state rotation vector when the disc is in exact rotational balance with no contribution from radial pressure support is given by  $\Omega_k(r)\hat{\mathbf{z}}$  with  $\Omega_k(r) = \Omega_k(r_0)(r_0/r)^{3/2}$  where  $r$  is radius in a cylindrical coordinate system and where  $r_0$  is a fiducial radius with corresponding rotation rate  $\Omega_k(r_0)$ . As a general rule for axisymmetric basic states, for every radial extremum in the PV gradient there can exist an azimuthally propagating Rossby wave radially localized around the corresponding radial extremum point. In the simplest configuration in which there is some action-at-a-distance and no possibility of critical layers (cf. Marcus *et al.* (2013)), instability can arise from the resonant interaction of two or more Rossby waves (Heifetz *et al.* 1999). For example, in RWI unstable discs with a single axisymmetric pressure bump, there are two oppositely signed extrema in the radial gradient-PV which can each support Rossby waves with wave speeds boosted by the local flow of the disc (Umurhan 2010). Resonant interaction and subsequent instability can occur when the Rossby waves move with equal and opposite directions in some reference frame. We also note that the previous studies by Mamatsashvili and Rice (2009) and Mamatsashvili and Chagelishvili (2007) have explored the PV profiles in order to understand the evolution of the dynamics in PP-disks,

This perspective of interacting Rossby waves and resonant instability has been extended to include other physical effects, e.g. the interaction of resonant Rossby-gravity waves (Harnik *et al.* 2008, Rabinovich *et al.* 2011) and Alfvén-Rossby waves in sheared MHD flows (Heifetz *et al.* 2015). Generally when there is some additional physical effect included in shear flows, baroclinicity begins to take on

<sup>1</sup>sometimes known in the astrophysical literature as “vortensity”

an important dynamical role. In the case of the above-mentioned Rossby-gravity waves, pressure and density isolines of the perturbed flow do not coincide (i.e.  $\nabla\rho \times \nabla p \neq 0$ ) and, consequently, a certain amount of vorticity creation can occur. Places in a zonal disc flow in which there is a stark density gradient can support vorticity waves<sup>2</sup> with this baroclinic “vorticity creation” effect and this, in turn, can influence the other vorticity waves of the physical system and consequently change the stability character of the whole physical arrangement. A physical perspective developed for a self-gravitating sheared system should therefore take this baroclinic effect into account.

Lovelace and Hohlfeld (2013) have extended the examination of the RWI to infinitely thin self-gravitating discs and they show that, among other things, transition to instability is very subtle and that, if the conditions are right, a localized pressure deficit (as opposed to a pressure bump) can also lead to instability. It is the aim of this work to begin constructing a potential vorticity view to use in understanding unstable transition in sheared self-gravitating configurations. Additionally, careful numerical experiments of self-gravitating disc systems performed by Lin and Papaloizou (2011a,b) examined the 3D disc response of a radial section of a disc supporting a gap cleared out by a perturbing planetary body. The numerical experiments show that the resulting azimuthally averaged radial PV of their discs exhibit complicated gradient-PV structure, especially in the near vicinity of the two gap walls. Interestingly, only one of the gap walls is dynamically stable while the other shows instability followed by restricted vortex roll up (see Figure 6 of Lin and Papaloizou (2011b)). The effort of this first of a series of studies is (also) motivated by making mechanistic and rational sense of these results.

The inclusion of self-gravity in any fluid examination is subtle business because it introduces a certain amount of ellipticity to the underlying PDE’s. In practice this means having to match the solutions developed within the interior of the fluid onto external solutions with appropriate far-field boundary conditions. To circumvent the difficulties inherent to this process and simplify the posed mathematical problem, most examinations of accretion disc dynamics in the astrophysical literature treat the disc as either (i) vertically thin infinitesimally speaking (“razor thin”) with a non-zero vertically integrated surface density or (ii) as a vertically infinite cylinder. The former is relatively tractable analytically as far field solutions are straightforward to construct while the latter considerably simplifies the system because boundary conditions are needed to be specified in the radial direction only and these, in turn, may be represented as parameters of the basic state. Of course, both have severe shortcomings if realism is the goal. However, as far as the action of localized edgewaves under the influence of self-gravity is concerned, using one or the other of the two approximations ought not contaminate the physical content of any result developed forthwith. The model analyzed in this study will be in the vertically infinite cylinder setting, which is basically a two-dimensional system and the simplest of the two analytically tractable possibilities.

In section 2 we formulate the equations of motion in this two-dimensional model setting and develop an evolution equation for the disc vorticity. We further develop both basic states and perturbations. Section 3 concentrates on dynamics where the basic state shows a radial jump in the vorticity and density fields. The ensuing interfacial vorticity wave dynamics are deconstructed into the dynamics as arising from standard Rossby edge wave dynamics and the concomitant presence of interfacial gravity waves (sections 3.1 and 3.2). The latter of these is further analyzed in terms of Boussinesq and non-Boussinesq contributions (sections 3.2.1 and 3.2.2). Most importantly, it is in the latter of these subsections where we identify the dynamics arising from self-gravity as being in the Boussinesq and non-Boussinesq category of responses. These deconstructed analysis elements are reunited in section 3.3 and shown how they effect the overall stability of the system. It is in this section we show how, indeed, self-gravity always has a stabilizing influence irrespective of the radial stratification of the basic state. In section 4 we discuss the results and their applications for further analysis.

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<sup>2</sup>Although vorticity waves and Rossby waves are often used interchangeably, throughout this manuscript we distinguish between them. We consider Rossby waves as associated with vorticity production by the advection of the mean shear, and vorticity waves as associated with any vorticity creation.

## 2. Formulation

### 2.1. Disc equations

We consider a 2D gaseous disc model whose radial ( $r$ ) and azimuthal ( $\theta$ ) momentum equation components are written as,

$$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial \phi}{\partial r} - \frac{G_{\text{grav}} M}{r^2}, \quad (1)$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{\partial \phi}{r \partial \theta}. \quad (2)$$

The dynamics is described in the inertial frame of the disc where  $u$  and  $v$  are the radial and azimuthal velocities of the flow. We have included in equation (1) a model Keplerian potential representing the gravitational force of an object with mass  $M_*$  located at  $r = 0$ . Our strategy is to remove this explicit potential and rewrite the equations as a deviation around this basic rotationally supported state. Thus the basic balance is between the full centripetal acceleration term  $v^2/r$  and the star's radial force. Thus we write  $v = v_k \equiv r\Omega_k$ , where  $v_k$  is the basic Keplerian flow, and we find that

$$\frac{v_k^2}{r} = \frac{G_{\text{grav}} M_*}{r^2}, \quad \implies \quad \Omega_k = \sqrt{\frac{G_{\text{grav}} M_*}{r^3}}. \quad (3)$$

We then rewrite the azimuthal velocity as this Keplerian base state plus a deviation flow,  $v \mapsto v_k + v$  where  $v$  is henceforth understood to be the deviation azimuthal velocity from this basic Keplerian state. As such, equations (1-2) are rewritten into the more convenient form as

$$\frac{Du}{Dt} - \left( \frac{v^2}{r} + 2\Omega_k v \right) = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial \phi}{\partial r}, \quad (4)$$

$$\frac{Dv}{Dt} + u \left( \frac{v}{r} + \frac{1}{r} \frac{\partial r^2 \Omega_k}{\partial r} \right) = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{\partial \phi}{r \partial \theta}, \quad (5)$$

where the Lagrangian time derivative is given as

$$\frac{D}{Dt} = \left( \frac{\partial}{\partial t} + (\mathbf{u} + \Omega_k r \hat{\theta}) \cdot \nabla \right) = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \left( \frac{v}{r} + \Omega_k \right) \frac{\partial}{\partial \theta}.$$

where  $\mathbf{u} \equiv u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}}$ . In this way, the rotationally supported state is built directly into these equations by having explicitly removed the potential arising from the central gravitating object.<sup>1</sup> When we refer to the rotation rate at some position  $r_0$  we write  $\Omega_k(r_0) \equiv \sqrt{\frac{G_{\text{grav}} M_*}{r_0^3}}$ . The disc surface gas density is  $\rho$ ,  $p$  represents the gas pressure, and  $\phi$  is the gravitational potential resulting from the gaseous material within the disc, satisfying the Poisson's relation:

$$\nabla^2 \phi = 4\pi G_{\text{grav}} \rho. \quad (6)$$

We assume further that the gas is incompressible, hence

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0, \quad (7)$$

<sup>1</sup>In an exact treatment of a three-dimensional system rotating around a central potential, it is a fundamental fact (e.g. Kippenhahn *et al.* (1990)) that if the equation of state is assumed to be barotropic then it follows that the basic rotational state  $\Omega_k$  will necessarily be independent of the vertical coordinate. The model potential considered in equations (1-2), while not exactly correct as far as a real disc is concerned, is meant to be a facsimile of the aforementioned fact.

and that, consequently, the local density of the gas is an advected passive tracer:

$$\frac{D\rho}{Dt} = 0. \quad (8)$$

Equations (4-8) form a complete set to describe the evolution of the disc variables  $(u, v, \rho, p, \phi)$ . In this paper however we focus on interfacial vorticity waves, thus we transform (4) and (5) into the vorticity equation:

$$\frac{D}{Dt} [(\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}}] + u \frac{\partial Q_{\text{kep}}}{\partial r} = \frac{1}{\rho^2} (\nabla \rho \times \nabla p) \cdot \hat{\mathbf{z}} \quad (9)$$

or more explicitly after moving the Coriolis-like term onto the righthand side of the above expression,

$$\frac{Dq}{Dt} = -u \frac{\partial Q_{\text{kep}}}{\partial r} - \frac{1}{r\rho^2} \left[ \frac{\partial \rho}{\partial \theta} \frac{\partial p}{\partial r} - \frac{\partial \rho}{\partial r} \frac{\partial p}{\partial \theta} \right], \quad (10)$$

where  $q \equiv \frac{1}{r} \frac{\partial}{\partial r} (rv) - \frac{1}{r} \frac{\partial u}{\partial \theta}$ , is the relative vorticity perpendicular to the disc plane. In arriving to equation (10) we have made explicit use of equation (7). it should be noted that as the flow is incompressible, there is essentially no difference between the vorticity and the vortensity equation. We observe that generation of vorticity  $q$ , arises from radial advection of the mean disc vorticity  $Q_{\text{kep}}$  associated with the Keplerian shear (Fig. 1(a)),

$$Q_{\text{kep}} \equiv \frac{1}{r} \frac{\partial (r^2 \Omega_k)}{\partial r}, \quad (11)$$

and the baroclinic torque  $\frac{1}{\rho^2} (\nabla \rho \times \nabla p) \cdot \hat{\mathbf{z}}$ , whose two terms generate shear and hence vorticity (Figs. 1(b)&1(c)). When isobars and isopicnals intersect, the same pressure gradient exerts larger acceleration on the gas whose density is smaller (simply because the pressure gradient force is  $-\frac{1}{\rho} \nabla p$ ).

## 2.2. Basic state and linearized dynamics

Equation set (4-8) admits an axisymmetric steady-state solutions (denoted by bars), satisfying the balance of forces:

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial r} = \frac{\partial \bar{\phi}}{\partial r} + \bar{v} \left( 2\Omega_k + \frac{\bar{v}}{r} \right), \quad (12)$$

with zero radial velocity ( $\bar{u} = 0$ ). This equation is supplemented by the solution of the Poisson equation in steady state

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \bar{\phi}}{\partial r} = 4\pi G_{\text{grav}} \bar{\rho}. \quad (13)$$

together with appropriate boundary conditions. The two basic state equations are characterized by four functions of radius,  $\bar{v}$ ,  $\bar{\rho}$ ,  $\bar{p}$  and  $\bar{\phi}$ , which means that the model as presented is unconstrained and, as a consequence, any basic state flow profile may be specified freely by two arbitrary functions of radius. As we are concerned with the dynamics localized primarily in the region of some radial position  $r_0$ , the values of particular basic state quantities evaluated there, like  $\bar{v}$  and  $\frac{\partial \bar{p}}{\partial r}$  (which turn out to be important in what follows), can be and are treated henceforth as parameters of the system.

Linearization of the vorticity equation (10) with respect to this basic state reveals:

$$\frac{D_L q'}{Dt} = -u' \frac{\partial}{\partial r} \left( \bar{q} + \frac{1}{r} \frac{\partial (r^2 \Omega_k)}{\partial r} \right) - \frac{1}{r\bar{\rho}^2} \left[ \frac{\partial \rho'}{\partial \theta} \frac{\partial \bar{p}}{\partial r} - \frac{\partial \bar{\rho}}{\partial r} \frac{\partial p'}{\partial \theta} \right], \quad (14)$$

where  $\bar{q} = \frac{1}{r} \frac{\partial}{\partial r} (r\bar{v})$  is the basic state vorticity, small perturbations from the basic state are denoted by primes, and  $\frac{D_L}{Dt} \equiv \frac{\partial}{\partial t} + \left( \frac{\bar{v}}{r} + \Omega_k \right) \frac{\partial}{\partial \theta}$  is the linearized Lagrangian time derivative. Hence, vorticity

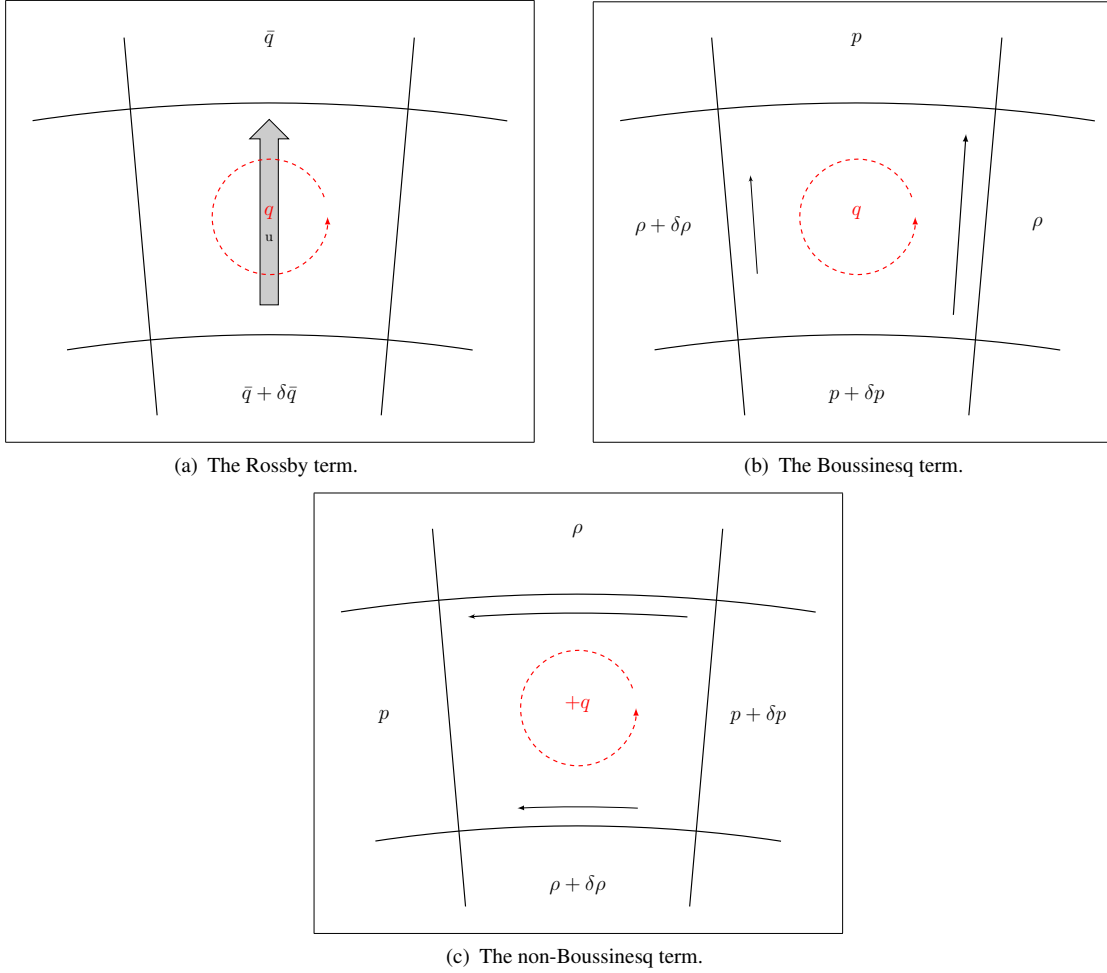


Figure 1. Mechanisms of vorticity generation  $q$  (indicated by a counterclockwise dashed circle) in cylindrical geometry. (a) Radial advection of the mean disc vorticity (accounted for the Rossby wave mechanism, Section 3.1). The thick arrow represents the radial velocity. (b&c) The two components of the baroclinic torque (accounted for the Boussinesq and non-Boussinesq wave mechanisms, respectively, Section 3.2,3). Black arrows indicate the direction and relative strength of the velocity induced by the torque. The order of the terms at the RHS of equation (7) corresponds to the order of the diagrams.

perturbation may be generated by radial advection of the mean ambient (Keplerian plus basic state) vorticity

$$\bar{Q} = \bar{q} + Q_{\text{kep}} = \bar{q} + \frac{1}{r} \frac{\partial(r^2 \Omega_k)}{\partial r},$$

and by the linearized action of the baroclinic torque, where its second term is neglected when assuming a Boussinesq flow. The baroclinic torque can be written in terms of the radial displacement perturbation  $\xi'$  ( $u' = \frac{D_L}{Dt} \xi'$ ) when noting that linearization of the incompressible continuity equation (8) implies:

$$\frac{D_L \rho'}{Dt} = -u' \frac{\partial \bar{\rho}}{\partial r} \Rightarrow \rho' = -\xi' \frac{\partial \bar{\rho}}{\partial r}, \quad (15)$$

that is to say that all small density perturbations result from radial advection of the basic state density. Substitute (15) into (14) gives:

$$\frac{D_L q'}{Dt} = -u' \frac{\partial \bar{Q}}{\partial r} + \frac{1}{r \bar{\rho}^2} \frac{\partial \bar{\rho}}{\partial r} \frac{\partial}{\partial \theta} \left[ \xi' \frac{\partial \bar{\rho}}{\partial r} + p' \right]. \quad (16)$$

In barotropic incompressible flow material conservation of density implies material conservation of

pressure and hence, similar to (15),  $p'_{\text{barotropic}} = -\xi' \frac{\partial \bar{p}}{\partial r}$ . Therefore, the sum of the terms in the squared brackets represent the residual baroclinic pressure perturbation.

### 3. Interfacial vorticity wave dynamics

As we mentioned in the Introduction, both numerical experiments and observations suggest that sharp gradients of vorticity and density emerge in different stages of the disc evolution. Such sharp gradients (either in density or vorticity) will support the generation of interfacial (edge) waves. Here we wish to focus on their basic mechanism of propagation and growth and, hence for simplicity, we assume strong gradients existing in both the density and vorticity only and replace the corresponding gradient terms appearing by  $\delta$ -functions, i.e.,

$$\frac{\partial \bar{Q}}{\partial r} = \Delta \bar{Q} \delta(r - r_0), \quad \frac{\partial \bar{p}}{\partial r} = \Delta \bar{p} \delta(r - r_0). \quad (17)$$

In practice we are assuming that there is no vorticity production anywhere except for  $r = r_0$ , This is a valid assumption only if  $\frac{\partial \bar{q}}{\partial r} \gg \frac{\partial Q_{\text{kep}}}{\partial r} = -\frac{3}{4} \sqrt{\frac{G_{\text{grav}} M_*}{r^5}}$ , therefore we expected that in these cases the dynamics can be reasonably represented by the interfacial waves approximation (cf. analogy to MHD shear flow Heifetz *et al.* (2015)).

Equation (14) implies then that the generated vorticity perturbations must have  $\delta$ -function structures as well. Thus for a wave-like solution the vorticity perturbation takes the form:

$$q'(r, \theta, t) = \hat{q}_0 \delta(r - r_0) e^{i(m\theta - \omega t)} \quad (18)$$

(where the zero subscript indicates evaluation at  $r_0$  and the azimuthal wavenumber  $m$  is taken to be positive). Since the gas is assumed incompressible, we can use (7) to introduce the perturbation streamfunction  $\psi'$  satisfying:

$$u' = -\frac{1}{r} \frac{\partial \psi'}{\partial \theta}; \quad v' = \frac{\partial \psi'}{\partial r} \quad (19)$$

so that  $q' = \nabla^2 \psi'$ . Thus in order for  $\psi'$  to satisfy (18), it should be in the form:

$$\psi' = \hat{q}_0 G(r, r_0, m) e^{i(m\theta - \omega t)}. \quad (20)$$

where the Green function  $G$  (not to be confused with the Gravitational constant denoted by  $G_{\text{grav}}$ ) is:

$$G(r, r_0, m) = -\frac{r_0}{2m} \begin{cases} \left(\frac{r}{r_0}\right)^{-m}, & \text{if } r \geq r_0 \\ \left(\frac{r}{r_0}\right)^m, & \text{if } r \leq r_0 \end{cases} \quad (21)$$

satisfying  $\nabla^2 G = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial G}{\partial r} \right) - \left(\frac{m}{r}\right)^2 G = \delta(r - r_0)$ , together with the boundary conditions  $G(r = 0, \infty) = 0$ . Note that while  $u'$  is continuous,  $v'$  is discontinuous at  $r_0$  and this infinite shear corresponds to the  $\delta$ -function of the vorticity perturbation there. An example for the structure of  $\psi'$  is shown in Fig. 2.

We assume a wavelike solution for all the perturbed values in the form of  $f' = \hat{f}(r, r_0, m) e^{i(m\theta - \omega t)}$ , but before solving the general dispersion relation for equation (16) we will split the problem to the dynamics arises from the existence of the mean vorticity and the mean density gradients.

#### 3.1. Interfacial Rossby waves

Only the basic state shear may account to sharp vorticity gradients. In the absence of mean density gradient equation (16), with the basic state of (17), becomes:

$$\frac{D_L q'}{Dt} = -u' \Delta \bar{Q} \delta(r - r_0), \quad (22)$$

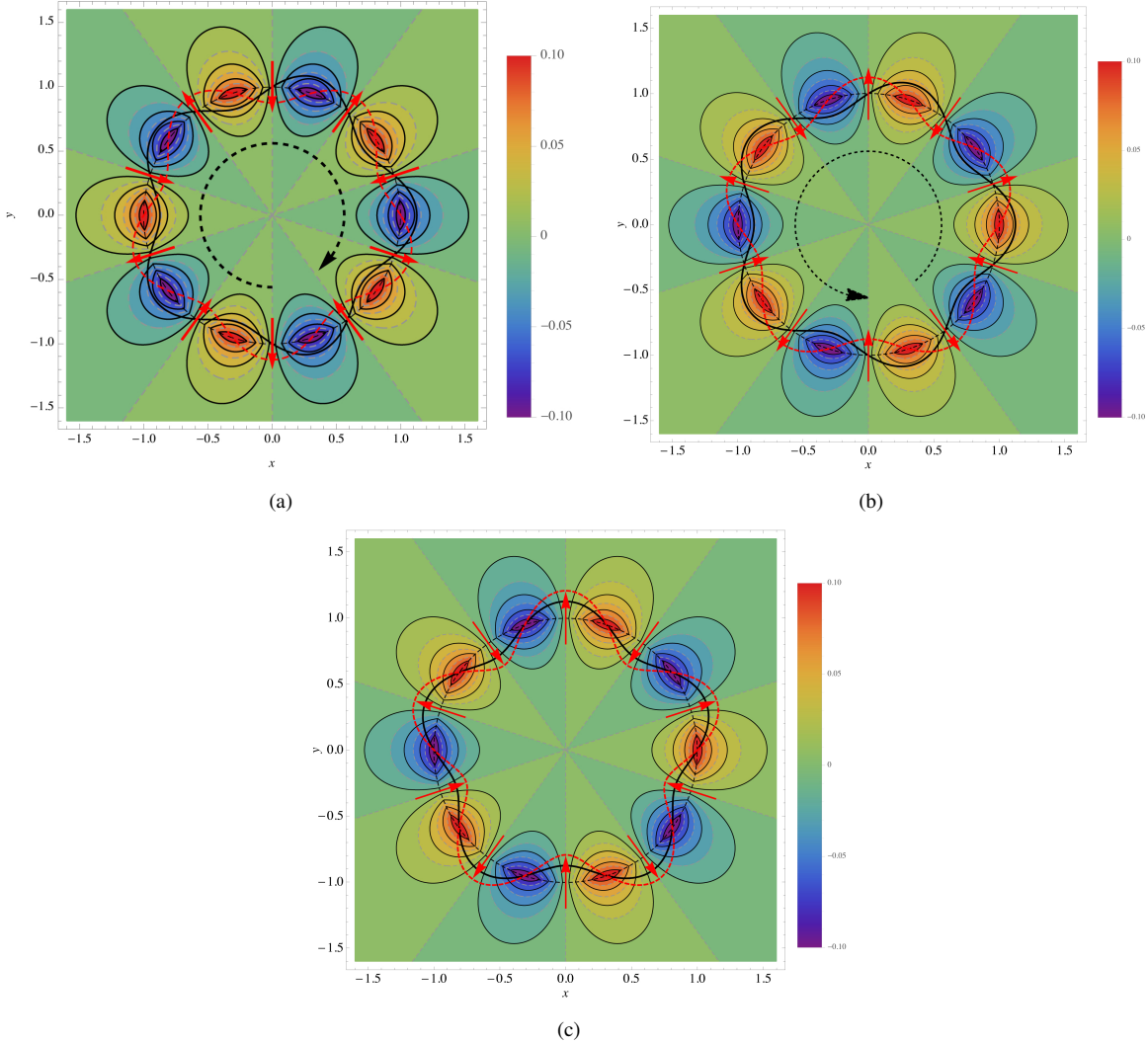


Figure 2. Contour plots of the streamfunction  $\psi'$  (16) for an azimuthal wavenumber  $m = 5$ , centered at the interface  $r_0 = 1$ . Positive values of  $\psi'$  correspond to negative values of  $q'$ , hence the red arrows indicate the direction of the radial perturbation velocity  $\frac{D_t \xi}{Dt} = u'$  corresponding to  $\psi'$ . The black wavy line represents the interface displacement  $\xi'$  at present time, while the red dashed one represents its evolution at some time  $t + \Delta t$  due to the advection of  $u'$ . Scenarios in which the vorticity and the displacement are (a) in phase resulting clockwise propagation (with respect to the mean flow) (b) in anti-phase resulting counterclockwise propagation (c) in quadrature resulting instability.

which together with (18-21), provides the dispersion relation:

$$\left( \frac{\omega_{Ros}}{m} - \Omega_0 \right) = \left( \frac{\Delta \bar{Q}}{2m} \right)_{r_0}, \quad \Omega_0 \equiv \frac{\bar{v}_0}{r_0} + \Omega_k(r_0), \quad (23)$$

where  $\Omega_0$  is the local rotation rate of the disc at position  $r_0$ . Equation (22) also implies that  $\hat{q}_0 = -\Delta \bar{Q} \xi_0$ , that is to say, that all variations in the vorticity result from advection of the mean vorticity (Fig. 1(a)) at the interface of  $r_0$ . The result is an interfacial Rossby wave which propagates (relative to the mean flow) in a direction that keeps the lower mean vorticity to its left. Hence if, for instance, the mean flow and the mean vorticity gradient are of opposite signs the waves propagate counter the mean flow. The mechanism of propagation is such that vorticity anomalies induce circulation that both undulates the interface and advects the mean vorticity to generate fresh vorticity anomalies in concert with the undulated interface (Fig. 3(a)&3(b)). These Rossby waves are the interfacial versions of the plane-like Rossby waves (Rossby *et al.* 1939) that are generic in the mid-latitude atmosphere due



to the meridional gradient of the Coriolis force there.

The dispersion relation of (23) indicates that the waves are neutral on their own. However two remote interfacial Rossby waves can generate instability (e.g. Baines and Mitsudera (1994), Heifetz *et al.* (1999)) by “action-at-a-distance” resonance if the necessary conditions for shear instability are satisfied, i.e., that each wave propagates counter the mean flow (Fjørtoft 1953) and that the radial mean vorticity gradients at the two interfaces are of opposite sign (Rayleigh 1880). The essence of this resonance mechanism is illustrated in (Fig. 4(b)) and has been applied in order to explain the RWI in Umurhan (2010). When self-gravity is included, instability will manifest itself along similar lines with the same main requirement - that there are at least two separate interfaces able to resonate with one another. Self-gravity as a dynamical influence only becomes relevant when there is at least one density gradient placed somewhere in the flow. The dynamical mechanisms resulting from such a single density gradient remains to be elucidated and this is what we do next.

### 3.2. Interfacial gravity waves

In the absence of mean vorticity gradient together with the basic state expressed in equation (17), equation (16) becomes:

$$\frac{D_L q'}{Dt} = \frac{\partial}{\partial \theta} \left[ \xi' \frac{\partial \bar{p}}{\partial r} + p' \right] \frac{\Delta \bar{p}}{r \bar{\rho}^2} \delta(r - r_0), \quad (24)$$

where  $\bar{\rho}_0$  is evaluated as the averaged mean density across the jump. We first obtain the solution for an interfacial gravity wave under the Boussinesq approximation which neglects the pressure perturbation in the squared brackets.

#### 3.2.1. Boussinesq waves

At the position  $r = r_0$  the cross-stream displacement equation  $u' = \frac{D_L}{Dt} \xi'$ , together with the vorticity equation there yields:

$$\hat{q}_0 = -2m \left( \frac{\omega}{m} - \Omega_0 \right) \hat{\xi}_0, \quad \left( \frac{\omega}{m} - \Omega_0 \right) \hat{q}_0 = - \left( \frac{\Delta \bar{p}}{r \bar{\rho}^2} \frac{\partial \bar{p}}{\partial r} \right)_{r_0} \hat{\xi}_0, \quad (25)$$

and the corresponding dispersion relation is:

$$\frac{\omega_{\text{Bous}}}{m} - \Omega_0 = \pm \left[ \frac{1}{2rm} \frac{\Delta \bar{p}}{\bar{\rho}} \left( \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial r} \right) \right]_{r_0}^{1/2}. \quad (26)$$

The wave is neutral when the signs of  $\Delta \bar{p}$  and  $\frac{\partial \bar{p}}{\partial r}$  agree. For instance, if the mean flow is relatively weak, (12) is close to hydrostatic balance and under stable stratification ( $\Delta \bar{p} < 0$ )  $\frac{\partial \bar{p}}{\partial r} < 0$ . The wave propagation mechanism for such scenario is illustrated in Figs. 3(c,d). For clockwise propagation relative to the mean flow,  $\left( \frac{\omega_{\text{Bous}}}{m} - \Omega_0 \right) < 0$ , equation set (25) determines that  $q'$  and  $\xi'$  are in phase, hence the radial velocity field (indicated by the radial arrows) induced by the vorticity perturbation shifts the displacement in the clockwise direction. Since  $\Delta \bar{p} < 0$ , (15) indicates that positive radial displacements corresponds to positive density anomalies, thus for this wave  $q'$  and  $\rho'$  are in phase as well. Now the Boussinesq part of the baroclinic torque results from the first term at the squared brackets of (14) and according to the mechanism described in Fig. 2(a) this will shift the vorticity anomalies clockwise, in concert with the radial displacement. For the counter-clockwise solution the vorticity and the displacement anomalies are in anti-phase where the displacement and the density anomalies remain in phase.

The unstable stratified case ( $\Delta \bar{p} > 0$  &  $\frac{\partial \bar{p}}{\partial r} < 0$ ) generates instability of the Rayleigh-Taylor type.

The wave is advected by the mean flow  $\bar{v}_0$  and grows with a rate given by

$$\left\{ \left| \frac{m}{2r} \frac{\Delta\bar{\rho}}{\bar{\rho}} \left( \frac{1}{\bar{\rho}} \frac{\partial\bar{\rho}}{\partial r} \right) \right| \right\}_{r_0}^{1/2}. \quad (27)$$

The perturbation potential vorticity  $q'$  is shifted by a quarter of a wavelength counter-clockwise to  $\xi'$  and  $\rho'$  and  $\xi'$  are in anti-phase (Fig. 4(a)). The resonance amplification results from the fact that in this configuration  $u'$  is in phase with  $\xi'$ , and the Boussinesq baroclinic torque is in phase with  $q'$ .

We thus see that both the propagation rate and the instability mechanism of Boussinesq gravity waves depend heavily on the sign of the mean radial pressure gradient force. Hence, it is crucial to include the contribution of the mean self gravity in (12) to the calculations, if the mean potential gravity gradient is comparable in magnitude to the Coriolis and the centrifugal forces.

### 3.2.2. Non-Boussinesq waves

Considering only the non-Boussinesq component of the baroclinic torque, equation (24) yields:

$$\frac{D_L q'}{Dt} = \frac{\partial p'}{\partial \theta} \frac{\Delta\bar{\rho}}{r\bar{\rho}^2} \delta(r-r_0) \implies \left( \frac{\omega}{m} - \Omega_0 \right) \hat{q}_0 = - \left( \frac{\Delta\bar{\rho}}{r\bar{\rho}^2} \right)_{r_0} \hat{p}_0. \quad (28)$$

In order to evaluate  $\hat{p}_0$  we first linearize the azimuthal component of the momentum equation (5) to obtain:

$$\frac{D_L v'}{Dt} + \bar{Q} u' = - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{p'}{\bar{\rho}} - \phi' \right), \quad (29)$$

and note that since  $v'$  is discontinuous and changes sign at  $r_0$ ,  $\hat{v}_0$  must remain zero at all times. Hence at  $r_0$  the wave satisfies the angular momentum balance between the radial flux of the absolute vorticity and the perturbation azimuthal torque:

$$r_0 \bar{Q}_0 u'_0 = - \frac{\partial}{\partial \theta} \left( \frac{p'}{\bar{\rho}} - \phi' \right)_{r_0}, \quad (30)$$

which with the aid of (19) becomes:

$$\hat{p}_0 = \bar{\rho}_0 (\bar{Q}_0 \hat{\psi}_0 + \hat{\phi}_0). \quad (31)$$

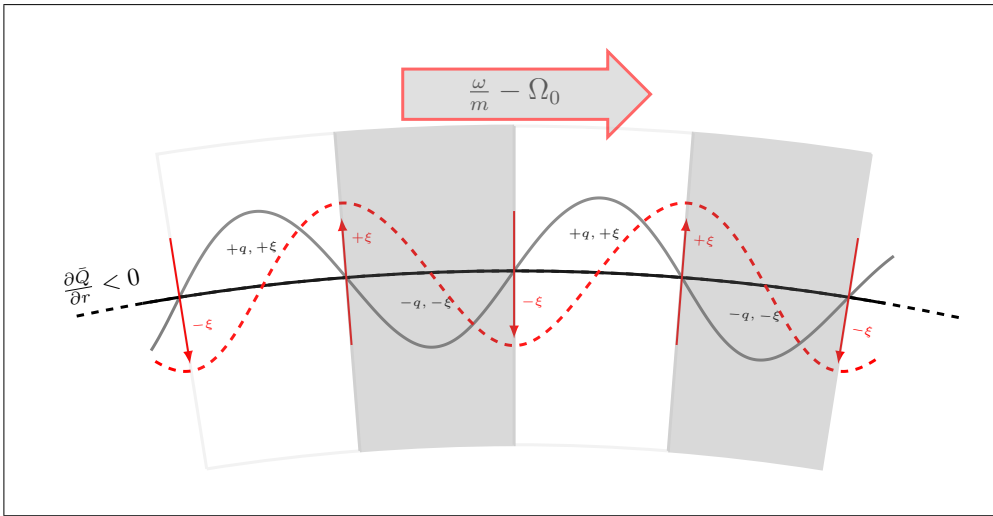
In the absence of self gravity perturbations ( $\hat{\phi}_0 = 0$ ), we can substitute  $\hat{\psi}_0$  directly in (24) to obtain, together with the displacement stream function relation, the dispersion relation:

$$\frac{\omega}{m} = \Omega_0 + \left( \frac{\Delta\bar{\rho}}{2m\bar{\rho}} \bar{Q} \right)_{r_0}. \quad (32)$$

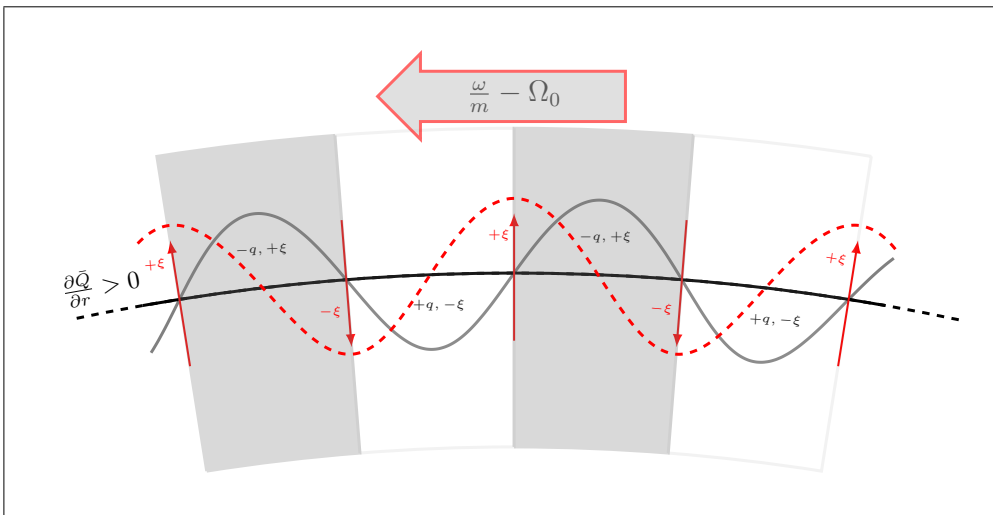
Consider for instance the stably stratified case where  $\Delta\bar{\rho} < 0$  and  $\bar{Q} > 0$  (the sign of the absolute vorticity matches the sign on the Kelperian one) so that the wave is neutral and propagates in the clockwise direction relative to the mean flow (Fig. 3(e)). Since the stream-function and the vorticity perturbations are in anti phase at  $r_0$ , equation (31) indicates that the pressure and the vorticity perturbations are in anti-phase there. Then the non-Boussinesq component of the baroclinic torque (proportional to  $-\frac{\partial\bar{\rho}}{\partial r} \frac{\partial p'}{\partial \theta}$ ) will generate fresh vorticity anomalies quarter of wavelength in the clockwise direction. Since the displacement  $\xi'$  should be translated in concert with the vorticity this implies that it should be in phase with the vorticity anomaly. We note that this wave cannot become unstable, even in unstably stratified setup. For counterclockwise propagation (Fig. 3(f))  $\Delta\bar{\rho} < 0$  and  $\bar{Q} < 0$   $\xi'$  is in anti phase with  $q'$  and  $p'$ .

In order to isolate the contribution of the self gravity perturbation we may assume a quasi-hydrostatic balance;  $\hat{p}_0 = \bar{\rho}_0 \hat{\phi}_0$ , in equation (31). Combining then (6) with (18) and the Green's

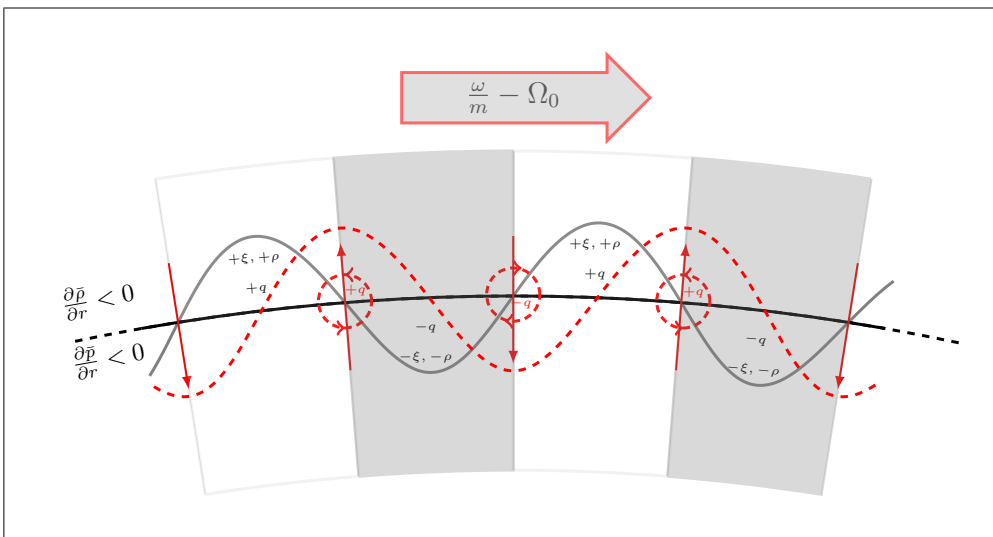
Rossby waves under the influence of self-gravity



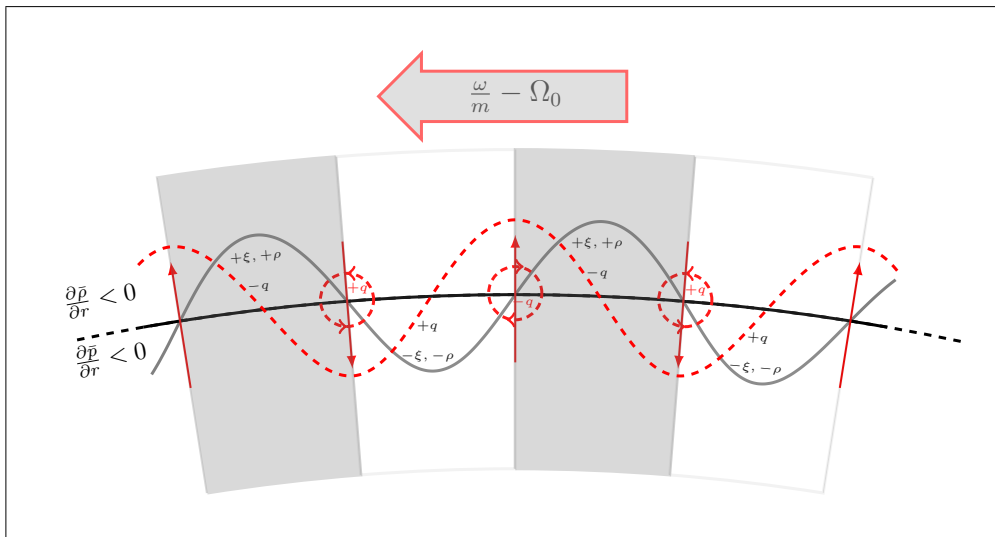
(a)



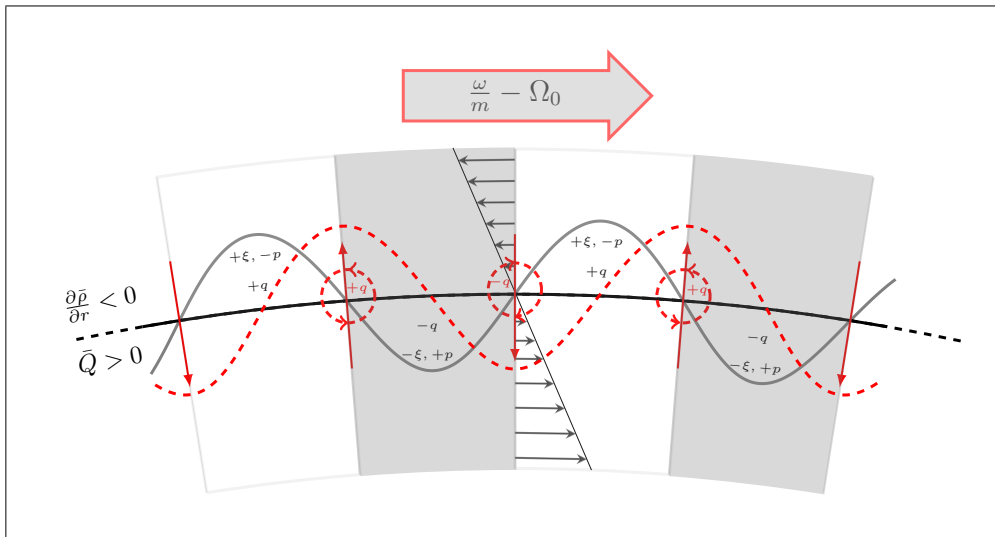
(b)



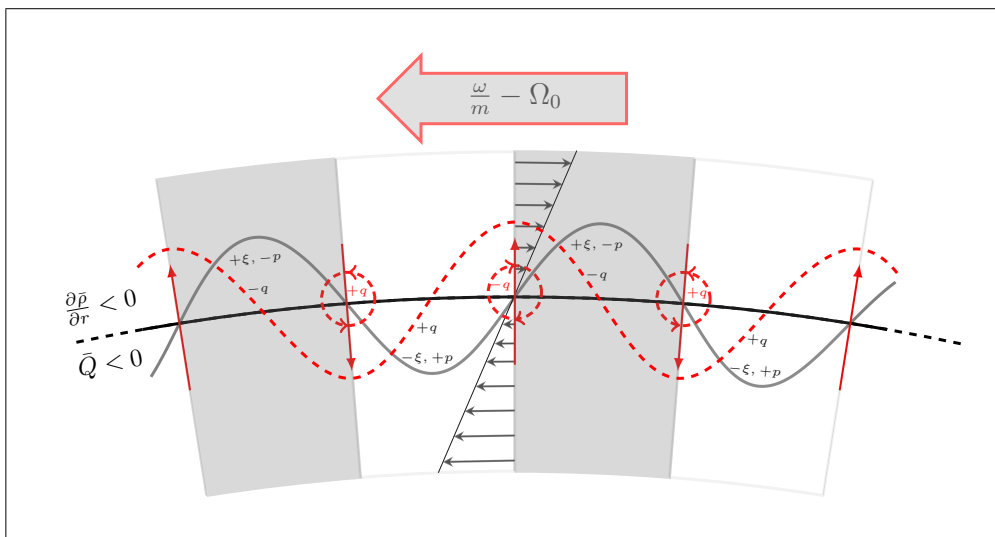
(c)



(d)



(e)



(f)

function solution gives  $\nabla^2 \phi' = -4\pi G_{grav} \Delta \bar{\rho} \delta(r - r_0) \xi'$ , hence

$$\begin{aligned} \phi' &= -4\pi G_{grav} \Delta \bar{\rho}_0 G(r, r_0, m) \hat{\xi}_0 e^{i(m\theta - \omega t)} \\ \Rightarrow \hat{p}_0 &= \bar{\rho}_0 \hat{\phi}_0 = \left( \frac{2\pi}{m} r_0 G_{grav} \bar{\rho}_0 \Delta \bar{\rho}_0 \right) \hat{\xi}_0. \end{aligned} \quad (33)$$

The last relation manifests the quasi-hydrostatic balance; if, for instance  $\Delta \bar{\rho}_0 < 0$ , an outward radial displacement shifts mass outward and therefore decreases both the gravitational potential and the pressure perturbation that balances the gravity fluctuation. Substitute (33) in (24), together with the generic relation obtained from the cross-stream displacement equation  $u' = \frac{D_t}{Dt} \xi'$ , we obtain:

$$\hat{q}_0 = -2m \left( \frac{\omega}{m} - \Omega_0 \right) \hat{\xi}_0, \quad \left( \frac{\omega}{m} - \Omega_0 \right) \hat{q}_0 = -\frac{2\pi}{m} G_{grav} \left[ \frac{(\Delta \bar{\rho})^2}{\bar{\rho}} \right]_{r_0} \hat{\xi}_0, \quad (34)$$

with the corresponding dispersion relation:

$$\frac{\omega}{m} = \Omega_0 \pm \left[ \frac{\pi}{m^2} G_{grav} \frac{(\Delta \bar{\rho})^2}{\bar{\rho}} \right]_{r_0}^{1/2}. \quad (35)$$

It is not surprising that quasi-hydrostatic fluctuations cannot generate instability. Consider again a relative clockwise propagation (Fig. 3(g)), then when  $\Delta \bar{\rho} < 0$  at  $r_0$ ,  $(q', \xi', p')$  are in phase with each other and in anti phase with  $(\phi', p')$  there, ensuring the coherent propagation of the wave. Even in the unstable stratified case ( $\Delta \bar{\rho}_0 > 0$ ), the wave remains neutral and propagates clockwise if  $(q', \xi', \phi', p')$  are in phase with each other and in anti phase with  $p'$  at  $r_0$ . For counterclockwise propagation (Fig. 3(h))  $\xi'$  is in anti phase with  $(q', \phi', p')$ .

### 3.3. Mixed Rossby-gravity interfacial waves

We can now combine the effects of all four components (Rossby, Boussinesq, the non-Boussineq vorticity flux and the non-Boussineq self gravity perturbation) to derive the general interfacial dispersion relation of (16), for the sharp gradients of equation (17). We apply the same strategy applied previously following all the substitutions we made in subsections (3.1,2), followed by simplifying equations (19-21) to derive an expression for  $\hat{u}_0$ , and further combining and simplifying equations (30, 32) to obtain an expression of  $\hat{p}_0$ . These two terms are both expressed in terms of the mean flow properties and  $(\hat{q}_0, \hat{\xi}_0)$ . This process results the generalized dispersion relation:

$$\begin{aligned} \left( \frac{\omega}{m} - \Omega_0 \right) &= \frac{1}{4m} \left( \Delta \bar{Q} + \frac{\Delta \bar{\rho}}{\bar{\rho}} \bar{Q} \right)_{r_0} \\ &\pm \left\{ \left[ \frac{1}{4m} \left( \Delta \bar{Q} + \frac{\Delta \bar{\rho}}{\bar{\rho}} \bar{Q} \right) \right]^2 + \frac{\Delta \bar{\rho}}{\bar{\rho}} \left[ \frac{1}{2mr} \left( \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial r} \right)_{r_0} + \frac{\pi G_{grav} \Delta \bar{\rho}}{m^2} \right] \right\}^{1/2}, \end{aligned} \quad (36)$$

together with the generic vorticity displacement ratio:

$$\hat{q}_0 = -2m \left( \frac{\omega}{m} - \Omega_0 \right) \hat{\xi}_0. \quad (37)$$

All the restoring mechanisms which, on their own, support neutral propagating waves now acting together to stabilize the only Rayleigh-Taylor like Boussinesq instability which is obtained when the product  $\Delta \bar{\rho} \cdot \frac{\partial \bar{p}}{\partial r} < 0$ . Therefore, the necessary criterion for instability is the same as for standard Rayleigh-Taylor instability

$$\left( \Delta \bar{\rho} \frac{\partial \bar{p}}{\partial r} \right)_{r_0} < 0.$$

With the inclusion of self-gravity, the sufficient condition for instability for a single edge wave is

$$\left( \Delta \bar{\rho} \frac{\partial \bar{p}}{\partial r} \right)_{r_0} < -\frac{2r_0}{m} \left[ \left( \frac{\bar{\rho} \Delta \bar{Q} + \Delta \bar{\rho} \bar{Q}}{4} \right)^2 + \pi G_{grav} \bar{\rho} \Delta \bar{\rho}^2 \right]_{r_0}. \quad (38)$$

The total composite expression within the square brackets on the right hand side of equation (38) is always greater than zero, thus the righthand side is always less than zero implying that these non-Boussinesq terms act in concert to stabilize the familiar Rayleigh-Taylor instability.

#### 4. Discussion

This single-interface setup and analysis has been meant to be one that showcases the propagation and instability of individual vorticity waves under the influence of mean vorticity and density gradients together with in the presence of self-gravity. The direction of propagation and the ability of the wave to be unstable boils down to the phasing relationship between the vorticity and the radial displacement anomalies. When the vorticity and the displacement anomalies are in phase (antiphase), the result will be a clockwise (anti-clockwise) propagating wave as in fig 2(a)(2(b)), when the displacement is in a quarter of a wavelength behind the vorticity, an instability will occur as in fig 2(c). It is not surprising that where there was only one edge wave present in the absence of gravitational dynamics, upon its inclusion (both through the global pressure gradient and self-gravity) an edge can support two so-called *kernel-gravity waves* (Harnik *et al.* 2008). It is also not surprising that one recovers the necessary condition for Rayleigh-Taylor instability if the flow is unstably stratified in the classical sense, i.e. if  $\Delta \bar{\rho} \frac{\partial \bar{p}}{\partial r} < 0$ .

However, what is surprising is the result embodied in the stability criterion found in equation (38): that in the single interface model kernel gravity waves are stabilized by self-gravity and its stabilizing role is unaffected by the stratification of the background state. One could easily imagine there being a location in a disc where the local background pressure gradient is identically zero (because of, say the global radial distribution of the gas material). In that case the effective radial component of gravity is zero and there ought not be any Rayleigh-Taylor like dynamics present. Yet, if the density jump across the interface is non-zero, the consequent activation of self-gravity works to stabilize disturbances and its dynamical effect is to alter the vorticity wave propagation speed only.

It is also worth noting the stabilizing effects of both self-gravity and the non-Boussinesq effects are diminished with increased azimuthal wavenumber. This is appeared consistent with intuition as the dynamical influence of self-gravity weakens the smaller the size of the dynamical object. As the dispersion relation in equation (36) shows, the non-Boussinesq and self-gravitational effects act to stabilize small  $m$  (long azimuthal wavelength) disturbances but leave the short wavelengths unaffected.

What we therefore conclude is that in a system with only one interface, instability occurs only if the flow is unstably stratified (Fig. 4(a)). As far as the edge waves are concerned, self-gravitational effects are always stabilizing. If the edge wave is not exponentially growing, self-gravity alters the speed of propagation of the two kernel gravity waves present along the individual interface. This naturally leads to the next spate of questions: what happens when well separated edgewaves (Fig. 4(b)), propagating along locally stably stratified regions, begin to interact with one another under the influence of both self-gravity and non-Boussinesq effects? How will what happens depend upon the Toomre Q-parameter? With respect to this, a form of the Q-parameter already appears in equation (36) and is given by

$$\mathcal{Q} \equiv \frac{\pi G_{grav} \Delta \bar{\rho}^2}{\bar{\rho} \Omega_0^2},$$

and any follow up study examining the interaction of separated interface waves will likely depend upon the magnitude of  $\mathcal{Q}$ . In this respect and as we alluded to earlier, previous studies show that sep-

arated edge waves will induce instability and cause roll-up of sheared flow provided the individually separated edgewaves become resonant with one another and there is some action-at-a-distance interaction possible. The resonance criterion requires the near matching of wave speeds of the individual edge waves. We have shown here that the propagation speeds of individual Rossby-gravity edgewaves will be modified by both the non-Boussinesq effects and by self-gravity. Our interest, in our follow up studies, is in how self-gravity influences the resonance mechanism as a function of the Toomre-like parameter  $\mathcal{Q}$ . Furthermore, with respect to the results of Lin & Papaloizou (2011a,b) we will seek as well to characterize and understand the conditions in which roll-up of disc-gap interfaces is expected to occur.

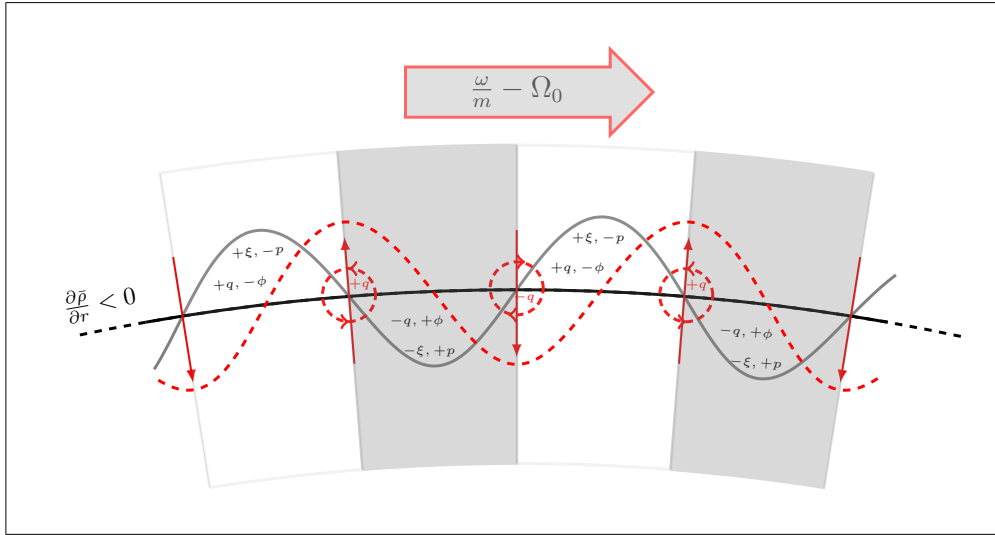
In addition we will explore how the dynamics is altered in cases which the continuous spectrum can not be neglected, the dynamics away from the interfaces are important and critical levels must be addressed.

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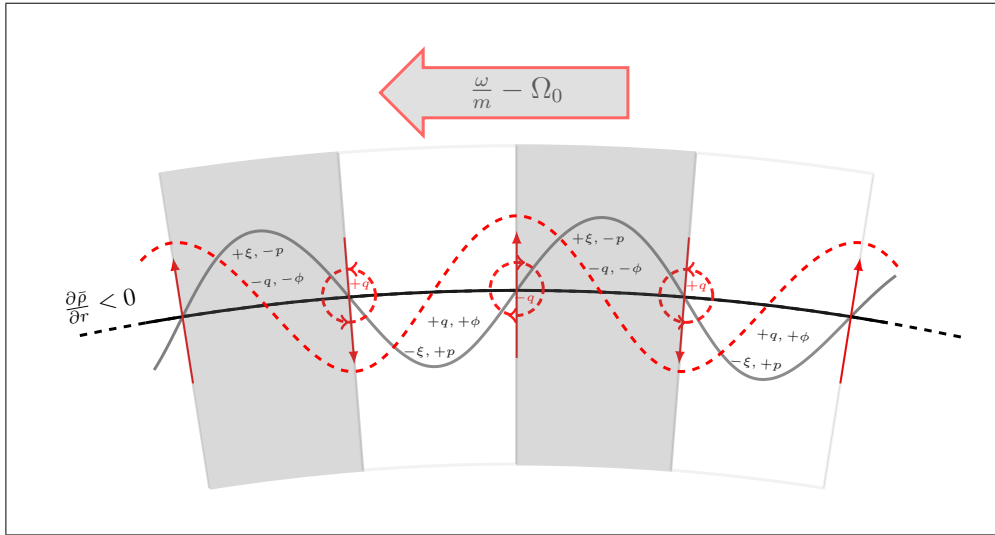
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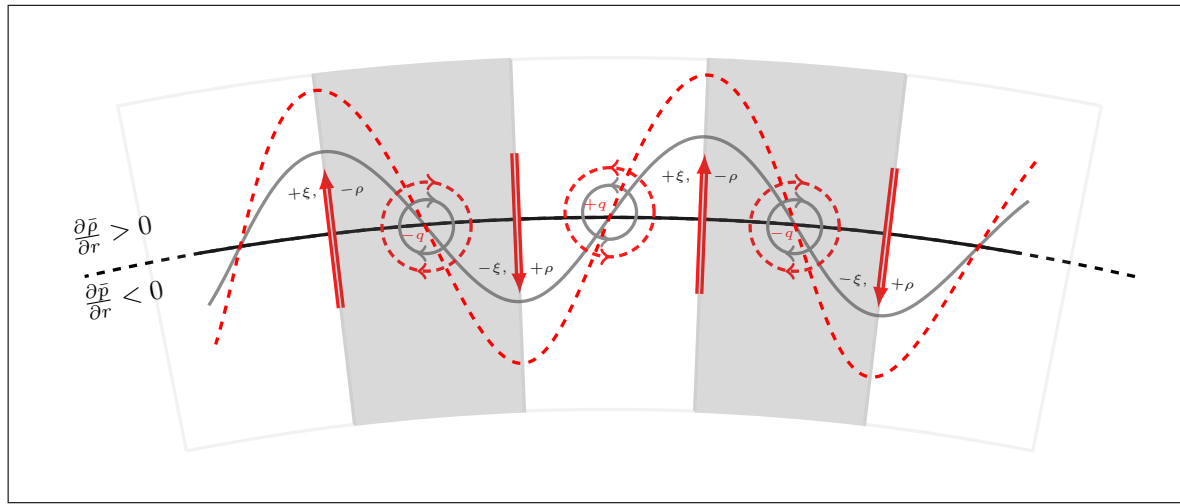


(g)

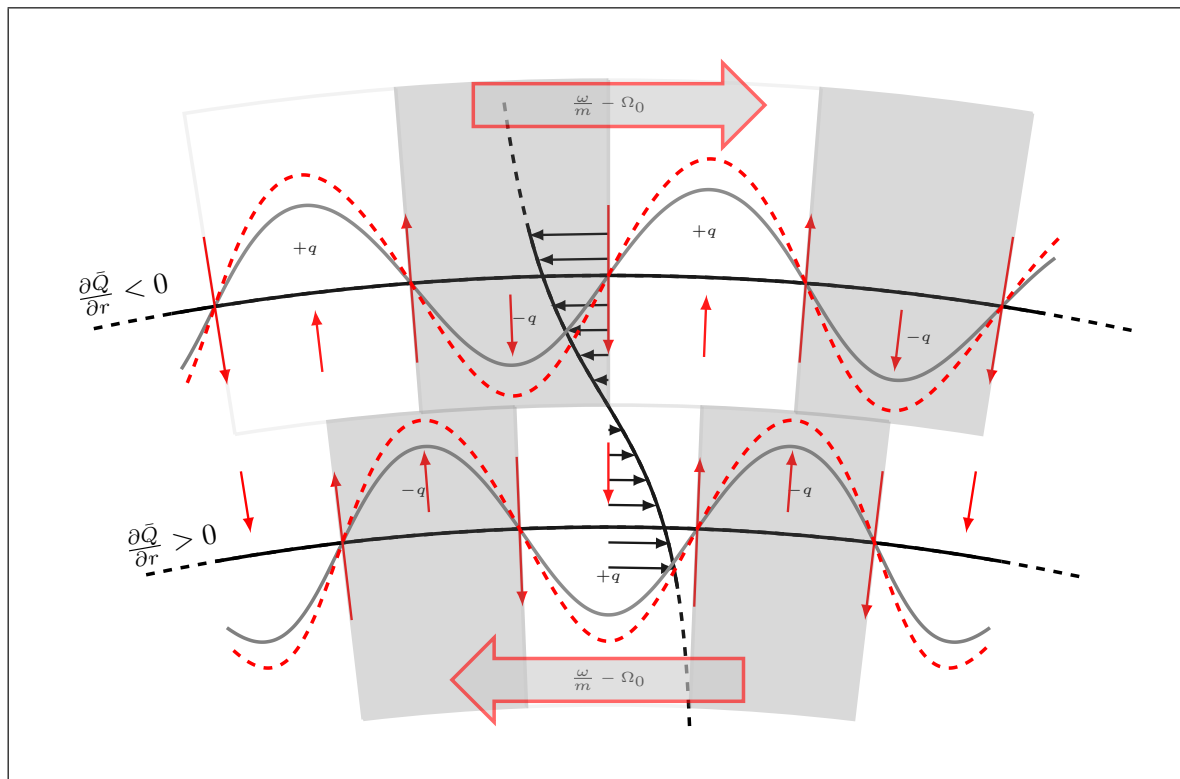


(h)

Figure 3. Vorticity waves propagation mechanisms. The black wavy lines represent present interfacial displacements while the red dashed ones represent their evolution (as in Fig. 2). On odd (even) letter figures the waves are propagating clockwise (counterclockwise), where each pair of clockwise and anti clockwise figure corresponds to a different wave propagation mechanism: (a,b) Rossby, (c,d) Boussinesq gravity, (e,f) Non-Boussinesq gravity, (g,h) Non-Boussinesq self gravitating waves. In (a)-(h) the background white (gray) color indicate positive (negative) signs of  $q'$ . The red dashed circular double head arrows indicate the vorticity production. More details about the different propagation mechanisms appear in Section 3.



(a) SG modified RayleighTaylor instability



(b) 2 interfacial Rossby waves action-at-a-distance instability

Figure 4. Instability mechanism of interfacial vorticity waves. (a) Single interface - when the vorticity (gray circles) and the displacement (wavy gray line) are in quadrature radial velocity induced by the vorticity (doubled red arrows) amplify the displacement (dashed red wavy line) and the baroclinic torque amplify back the vorticity (red dashed circles) resulting in positive feedback. (b) An illustration of a shear flow which can support two counter-propagating vorticity wave resonance. Lines and arrows are as in the previous figures. Each wave propagates counter its local mean flow and therefore the waves may become phase locked. Furthermore, the radial velocity induced by each wave on the opposed one (indicated by the smaller evanescent red arrows) amplify the other wave's amplitude. As a result continuous mutual amplification of the waves' amplitudes is obtained.