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# Notes and Correspondence

# Higher-order corrections for Rossby waves in a zonal channel on the $\beta$-plane

E. Heifetz,[a]* N. Paldor,[b] Y. Oreg,[c] A. Stern[c] and I. Merksamer[a]
[a] *Department of Geophysics and Planetary Sciences, Tel–Aviv University, Israel*
[b] *Ring Department of Atmospheric Sciences, The Hebrew University of Jerusalem, Israel*
[c] *Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot, Israel*

**ABSTRACT:** A formal analytic perturbation expansion in the $\beta$ term is carried out for the Rossby wave solution of the shallow-water equations in a zonal channel on the $\beta$-plane. Apart from a quantization of the meridional wave number, the presence of zonal boundaries alters, to first order, both the velocity and the geopotential structures of the wave but does not alter the phase speed of the wave. The ageostrophic component of the velocity field is identical in first order with that of the unbounded $\beta$-plane and is therefore not related to the presence of boundaries. In contrast, the first-order correction to the geostrophic velocity component is inherently related to the presence of walls as it ensures the vanishing of the total meridional velocity on the boundaries. This first-order correction to the geostrophic field yields only a third-order correction in the Rossby phase speed, as can be expected from symmetry considerations. Copyright © 2007 Royal Meteorological Society

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## 1. Introduction

The fundamental derivation of Rossby wave propagation on a $\beta$-plane was presented originally by Rossby (1939), and has been quoted since in many textbooks, e.g. Gill (1982), Pedlosky (1987), Holton (1992), Cushman-Roisin (1994). In a shallow-water framework, the dispersion relation of Rossby waves of the form

$$\exp\{\mathrm{i}(k_x x - \omega_{\mathrm{Ro}} t)\} \exp \mathrm{i} k_y y,$$

becomes (e.g. Gill, 1982)

$$\omega_{\mathrm{Ro}} = -\frac{\beta k_x}{k_x^2 + k_y^2 + (1/R_{\mathrm{d}})^2}. \tag{1}$$

(Here $(x, y)$ denotes the Cartesian zonal and meridional directions, $\mathbf{K} = (k_x, k_y)$, is the total wave number and $R_{\mathrm{d}} = \sqrt{gH}/f_0$ is the Rossby radius of deformation where $g$ is gravity and $H$ is the averaged thickness of the layer of fluid. Furthermore, $f_0 = 2\Omega \sin\phi_0$ is the Coriolis parameter at a central latitude $\phi_0$ where $\Omega$ is the Earth's angular velocity, $\beta = 2\Omega\cos\phi_0/R$ is the latitudinal derivative of the Coriolis frequency, evaluated at $\phi_0$, and $R$ is the radius of the Earth.)

This elegant result has been derived for an infinite $\beta$-plane. As such, it is inconsistent with the approximation of the $\beta$-plane (where $\beta = \mathrm{d}f/\mathrm{d}y$ is constant) which is valid only as long as $\beta y \ll f_0$ (i.e. $y \ll f_0/\beta = \tan\phi_0 R$). Hence, a more consistent set-up (although not necessarily realistic) for Rossby wave propagation on the $\beta$-plane is a channel whose zonal walls are located at $y = \pm L$. Such a channel configuration has been widely used in basic theoretical models for both baroclinic and and barotropic instability on a $\beta$-plane (e.g. Kuo, 1949, 1973; Charney, 1947; Philips, 1954; Howard and Drazin, 1964). For rotating annulus experiments which generate topographic Rossby waves, a radial $\beta$-plane approximation can be assumed where the radial velocity must vanish on the inner and outer concentric boundaries (e.g. Solomon *et al.*, 1993; Songnian *et al.*, 2001).

On an unbounded $\beta$-plane, the structure of the wave consists of geostrophic (denoted by the subscript 'g') and ageostrophic (denoted by the subscript 'a') velocity components, where the latter is determined by the former:

$$\mathbf{u}_{\mathrm{g}} = \frac{g}{f_0}\mathbf{z} \times \nabla h, \tag{2a}$$

$$\mathbf{u}_{\mathrm{a}} = \frac{1}{f_0}(\mathrm{i}\omega_{\mathrm{Ro}}\mathbf{u}_{\mathrm{g}} \times \mathbf{z} - \beta y \mathbf{u}_{\mathrm{g}}). \tag{2b}$$

($\mathbf{z}$ is the vertical unit vector and $h$ denotes the time- and space-dependent thickness of the fluid.) The presence of

* Correspondence to: E. Heifetz, Department of Geophysics and Planetary Sciences, Tel–Aviv University, Israel.
E-mail: eyalh@cyclone.tau.ac.il

boundaries requires a straightforward quantization of the meridional wave number to enable the vanishing of the geostrophic meridional velocity there, $v_{\mathrm{g}}(y=\pm L)=0$, i.e.

$$v_{\mathrm{g}}=\widehat{v}_0 \mathrm{e}^{\mathrm{i}(k_x x-\omega_{\mathrm{Ro}} t)} \cos k_y y, \tag{3a}$$

$$k_y=\frac{(2n+1)}{L}\frac{\pi}{2}, \tag{3b}$$

where $n$ is an integer, and only symmetric solutions around $y=0$ are considered. This quantization determines also the structure of the ageostrophic velocity, (2b), since (2a) and (3) suggest that

$$h_0=\widehat{h}_0 \mathrm{e}^{\mathrm{i}(k_x x-\omega_{\mathrm{Ro}} t)} \cos k_y y, \tag{4a}$$

$$\widehat{h}_0=-\frac{\mathrm{i} f_0}{k_x g}\widehat{v}_0, \tag{4b}$$

and thus the $O(\beta)$ ageostrophic meridional velocity, whose component becomes:

$$\begin{aligned} v_{\mathrm{a}} &= -f_0^{-1}(\mathrm{i}\omega_{\mathrm{Ro}} u_{\mathrm{g}}+\beta y v_{\mathrm{g}}) \\ &= -f_0^{-1}\left(\omega_{\mathrm{Ro}}\frac{k_y}{k_x}\sin k_y y+\beta y\cos k_y y\right)\widehat{v}_0 \mathrm{e}^{\mathrm{i}(k_x x-\omega_{\mathrm{Ro}} t)}, \end{aligned} \tag{5}$$

does not vanish at the walls. Therefore, the total velocity $\mathbf{u}=\mathbf{u}_{\mathrm{g}}+\mathbf{u}_{\mathrm{a}}$, does not satisfy the boundary conditions

$$v(y=\pm L)=v_{\mathrm{g}}(y=\pm L)+v_{\mathrm{a}}(y=\pm L)=0,$$

which implies that quantization of the meridional wave number by itself is insufficient for describing Rossby waves in a zonal channel. Hence, even without baroclinic or barotropic shear, the basic structure and frequency of Rossby waves on a $\beta$-plane might be altered due to the presence of zonal boundaries. Thus, in the next section we suggest a more rigorous perturbation scheme for the next order correction to the Rossby waves in a channel that satisfies the no-normal-flow boundary conditions at the walls of a channel on the $\beta$-plane.

## 2. Rossby waves in a zonal channel on the $\beta$-plane

The linearized shallow-water equations on the $\beta$-plane are (e.g. Gill, 1982):

$$\dot{u}=(f_0+\beta y)v-g\frac{\partial h}{\partial x}, \tag{6a}$$

$$\dot{v}=-(f_0+\beta y)u-g\frac{\partial h}{\partial y}, \tag{6b}$$

$$\dot{h}=-H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tag{6c}$$

where the dot implies the partial time-derivative. Standard quasi-geostrophic approximation of (6) leads then to (1) and (2) for a wavelike solution.

Denoting the small parameter $\widetilde{\beta}=\beta L/f_0=\cot\phi_0 L/R \ll 1$ (where the last inequality is required by the expansion of $f(y)$ only to first order in $y$ throughout the entire channel), then a perturbation expansion in $\widetilde{\beta}$ of the eigenfunctions and eigenfrequency of a zonal wave of the form $\exp\{\mathrm{i}(k_x x-\omega t)\}$, can be written as

$$\begin{pmatrix}\mathbf{u}\\ h\\ \omega\end{pmatrix}=\begin{pmatrix}\mathbf{u}_0\\ h_0\\ \omega_0\end{pmatrix}+\widetilde{\beta}\begin{pmatrix}\mathbf{u}_1\\ h_1\\ \omega_1\end{pmatrix}+\widetilde{\beta}^2\begin{pmatrix}\mathbf{u}_2\\ h_2\\ \omega_2\end{pmatrix}+\cdots \tag{7}$$

From the following symmetry argument we expect that the eigenfrequency $\omega$ should be odd in $\beta$, i.e. $\omega_0=\omega_2=\ldots=0$. The system (6) is invariant to a reflection (change of sign) of $y$, $v$, $h$, $\beta$ and $\omega$ (i.e. $t$). Since the variables $y$, $v(y)$ and $h(y)$ cannot appear in the expression for the eigenvalue $\omega$, it must have the same symmetry to reflection as $\beta$, which implies that only odd powers of $\beta$ appear. Notwithstanding this argument, we leave $\omega_0$ and $\omega_2$ in the derivation below and explicitly show that they vanish.

Then the zero-order terms of (6) yields

$$\begin{pmatrix}-\mathrm{i}\omega_0 & -f_0 & \mathrm{i}k_x g\\ f_0 & -\mathrm{i}\omega_0 & g\frac{\partial}{\partial y}\\ \mathrm{i}k_x & \frac{\partial}{\partial y} & -\mathrm{i}\omega_0/H\end{pmatrix}\begin{pmatrix}u_0\\ v_0\\ h_0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}, \tag{8}$$

where two of the eigensolutions are Poincaré plane waves, whose eigenfrequencies and eigenfunctions are:

$$\omega_{0\mathrm{Po}}^2=f_0^2+gHK^2, \tag{9a}$$

$$(\nabla\times\mathbf{u}_0)\cdot\widehat{\mathbf{z}}=f_0 h_0, \tag{9b}$$

$$K^2=k_x^2+k_y^2, \tag{9c}$$

and the third (zero frequency) solution yields the steady, geostrophic, solution:

$$\omega_0=0;\ \mathbf{u}_0=\left(\frac{g}{f_0}\right)\widehat{\mathbf{z}}\times\nabla h_0. \tag{10a, b}$$

We focus now on the higher-order corrections (in $\widetilde{\beta}$) to the geostrophic solution, (10), in the presence of walls. Taking the zonal derivative of (6b), subtracting it from the meridional derivative of (6a) and using (6c) we obtain the potential vorticity equation on the $\beta$-plane:

$$\frac{\partial}{\partial t}\left\{\frac{1}{f_0}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-\left(1+\widetilde{\beta}\frac{y}{L}\right)\frac{h}{H}\right\}=-\widetilde{\beta}\frac{v}{L}, \tag{11}$$

whose first-order term in $\widetilde{\beta}$, yields $\omega=\widetilde{\beta}\omega_1=\omega_{\mathrm{Ro}}$, where $\omega_{\mathrm{Ro}}$ is defined in (1). In order to find the first-order structure of the bounded Rossby waves, we take the second-order term, $O(\widetilde{\beta}^2)$, of (11):

$$\begin{aligned} &\omega_2\left\{\frac{1}{f_0}\left(\frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y}\right)-\frac{h_0}{H}\right\} \\ &+\omega_1\left\{\frac{1}{f_0}\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)-\frac{h_1}{H}-\frac{y}{L}\frac{h_0}{H}\right\}+\mathrm{i}\frac{v_1}{L}=0. \end{aligned} \tag{12}$$



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Using (1) and $\widetilde{\beta}\omega_1 = \omega_{\mathrm{Ro}}$ together with (10b) we get from (12):

$$\frac{\omega_2}{\omega_1}\frac{g}{f_0}k_x h_0 + \omega_1 L\left\{\frac{1}{f_0}\left(\mathrm{i}k_x v_1 - \frac{\partial u_1}{\partial y}\right) - \frac{h_1}{H} - \frac{y}{L}\frac{h_0}{H}\right\} + \mathrm{i}v_1 = 0. \tag{13}$$

Next we wish to find an equation for $v_1$ as a function of $v_0$ only. Towards this end, we first take the $O(\widetilde{\beta})$ term of (6a) which together with (10) yields:

$$h_1 = -\frac{\mathrm{i}f_0}{gk_x}v_1 + \left(\frac{y}{L} - \mathrm{i}\frac{k_y}{k_x}\frac{\omega_1}{f_0}\right)h_0. \tag{14}$$

The meridional derivative of the $O(\widetilde{\beta})$ term of (6c) yields:

$$\frac{\partial u_1}{\partial y} = \frac{\mathrm{i}}{\mathrm{k_x}}\left\{k_y\omega_1\frac{h_0}{H} + \frac{\partial^2 v_1}{\partial y^2}\right\}. \tag{15}$$

Now, substituting $h_1$ and $\partial u_1/\partial y$ back into (13) and writing $h_0 = -(\mathrm{i}f_0/gk_x)v_0$ (cf. 10b), we finally obtain a linear inhomogeneous differential equation for $v_1(y)$ where the inhomogeneous term is proportional to $v_0(y)$:

$$\left(\frac{\partial^2}{\partial y^2} + k_y^2\right)v_1 = \frac{1}{L}\left\{2\frac{f_0^2}{gH}y - k_x f_0\frac{\omega_2}{\omega_1^2}\right\}v_0. \tag{16}$$

The zero-order solution for $v_0(y)$ that satisfies the boundary conditions at $y = \pm L$ is given by (3a) with $k_y$ quantized as in (3b). The fact that $v_0(y)$ is a solution of the homogeneous equation associated with (16), together with the presence of a $yv_0(y)$ term in the inhomogeneous part of (16), implies that the general solution of $v_1$ is

$$v_1 = \left\{(ay^2 + by + c)\cos k_y y + (dy^2 + ey + f)\sin k_y y\right\} \times \widehat{v}_0 \mathrm{e}^{\mathrm{i}(k_x x - \omega t)}, \tag{17}$$

where $(a, \ldots, f)$ are some yet undetermined constants. The coefficient $c$ maybe absorbed in the zero-order solution and we therefore set it to zero. Evaluating the left-hand side of (16) for this form of $v_1(y)$ yields

$$\left(\frac{\partial^2}{\partial y^2} + k_y^2\right)v_1 = \left\{4k_y y(d\cos k_y y - a\sin k_y y) + 2(a + k_y e)\cos k_y y + 2(d - k_y b)\sin k_y y\right\}\widehat{v}_0\mathrm{e}^{\mathrm{i}(k_x x - \omega t)}. \tag{18}$$

Substituting (18) in the left-hand side of (16) fixes the coefficients $a, b, d, e$. Since $a = 0$, $e \propto \omega_2$. The boundary conditions $v_1(y = \pm L) = 0$ fix $e$ and $f$. Specifically $e = 0$ and therefore $\omega_2 = 0$ as expected. We finally obtain

$$v_1 = \frac{1}{2Lk_y^2}\frac{f_0^2}{gH}\left\{y\cos k_y y + k_y(y^2 - L^2)\sin k_y y\right\} \times \widehat{v}_0\mathrm{e}^{\mathrm{i}(k_x x - \omega t)}. \tag{19}$$

Hence, while for this order of $\widetilde{\beta}$ the boundaries do not affect the phase speed of the Rossby wave, they do alter the eigenfunction by adding first-order terms in $\widetilde{\beta}$ to the eigenfunctions. In order to express the first-order correction $(u_1, v_1, h_1)$ in terms of the zero-order height (stream function),

$$h_0 = \widehat{h}_0\cos(k_y y)\exp\{\mathrm{i}(k_x x - \omega_{\mathrm{Ro}}t)\},$$

we use the $O(\widetilde{\beta})$ terms of (6b), (6c) and (14) to get

$$u_1 = \frac{f_0}{2L}\left[\left\{2\frac{L\omega_1}{k_x f_0} - k_y^{-2} - (y^2 - L^2)\right\}\cos k_y y - \frac{y}{k_y}\sin k_y y\right]\widehat{h}_0\mathrm{e}^{\mathrm{i}(k_x x - \omega_{\mathrm{Ro}}t)}, \tag{20a}$$

$$v_1 = \frac{\mathrm{i}f_0 k_x}{2Lk_y^2}\left\{y\cos k_y y + k_y(y^2 - L^2)\sin k_y y\right\} \times \widehat{h}_0\mathrm{e}^{\mathrm{i}(k_x x - \omega_{\mathrm{Ro}}t)}, \tag{20b}$$

$$h_1 = \left\{\frac{y}{L}\left(1 + \frac{f_0^2}{gH}\frac{1}{2k_y^2}\right)\cos k_y y + \left(\frac{f_0^2}{gH}\frac{y^2 - L^2}{2Lk_y} + \frac{k_y\omega_1}{k_x f_0}\right)\sin k_y y\right\}\widehat{h}_0\mathrm{e}^{\mathrm{i}(k_x x - \omega_{\mathrm{Ro}}t)}. \tag{20c}$$

We now wish to understand why the modification in the eigenfunctions of $(\mathbf{u}, h)$, induced by the zonal boundary conditions, is not accompanied by changes in the Rossby wave frequency, $\omega_{\mathrm{Ro}}$. Hence, decomposing the first-order velocity into its geostrophic and ageostrophic components, i.e.

$$\mathbf{u}_1 = \mathbf{u}_{1\mathrm{a}} + \mathbf{u}_{1\mathrm{g}}, \quad \text{where } \mathbf{u}_{1\mathrm{g}} = \frac{g}{f_0}\mathbf{z} \times \nabla h_1,$$

then the first-order terms in (6) become

$$-\mathrm{i}\omega_1 u_0 = f_0\left(v_{1\mathrm{a}} + \frac{y}{L}v_0\right), \tag{21a}$$

$$\mathrm{i}\omega_1 v_0 = f_0\left(u_{1\mathrm{a}} + \frac{y}{L}u_0\right), \tag{21b}$$

$$-\mathrm{i}\omega_1 h_0 = -H\left(\frac{\partial u_{1\mathrm{a}}}{\partial x} + \frac{\partial v_{1\mathrm{a}}}{\partial y}\right). \tag{21c}$$

Subtracting the geostrophic component $(u_{1\mathrm{g}}, v_{1\mathrm{g}})$ from (20a,b), respectively, we find that

$$u_{1\mathrm{a}} = -\frac{g}{f_0^2}\left[k_x\omega_1\cos k_y y + \frac{f_0}{L}k_y y\sin k_y y\right]\widehat{h}_0\mathrm{e}^{\mathrm{i}(k_x x - \omega_{\mathrm{Ro}}t)}, \tag{22a}$$

$$v_{1\mathrm{a}} = -\mathrm{i}\frac{g}{f_0^2}\left[k_y\omega_1\sin k_y y + \frac{f_0}{L}k_x y\cos k_y y\right]\widehat{h}_0\mathrm{e}^{\mathrm{i}(k_x x - \omega_{\mathrm{Ro}}t)}. \tag{22b}$$



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Hence $\mathbf{u}_{1a} = f_0^{-1}(i\omega_{Ro}\mathbf{u}_{g0} \times \mathbf{z} - \beta y \mathbf{u}_{g0})$ exactly equals the first-order ageostrophic component of the unbounded Rossby waves given in (2b). Since (21) indicates that only the ageostrophic component of the first-order eigenfunction is involved in the Rossby wave dynamics, the role of the geostrophic first-order component is 'only' to ensure the vanishing of the total meridional velocity component on the boundaries.

We finally wish to examine how the boundaries alter the structure of the velocity and the geopotential distribution of the Rossby wave. In Figures 1 and 2, we plot some of the the velocity and geopotential fields for typical values of $f_0 = 10^{-4}\mathrm{s}^{-1}$, $g = 10\ \mathrm{ms}^{-2}$, $H = 10$ km, $L = 10^6$ m, and $(k_x, k_y) = \pi/2L(1,3)$ which implies $\widetilde{\beta} \approx 0.165$. In Figure 1 the first-order ageostrophic velocity vector field $\mathbf{u}_{1a}$, as obtained from (2b) or (22), versus the zero-order geopotential anomaly $h_0$, as obtained from (4), are plotted. Positive and negative values of $h_0$ are indicated respectively by solid and dashed contours. The structures of both $\mathbf{u}_{1a}$ and $h_0$ remain unchanged for both the unbounded and bounded $\beta$-plane configurations. Hence, Figure 1 represents the classical wave structure obtained by Rossby (1939). It is clear both from the plot and from (5) that the ageostrophic meridional velocity $v_{1a}$ does not vanish at the channels walls ($y = \pm L$).

While (21c) indicates that the divergence field is located $\pi/2$ out of phase to the west of the negative zero-order geopotential anomaly $h_0$, it is difficult, however, to deduce the divergence field of $\mathbf{u}_a$, directly from Figure 1. The reason for this is that $\mathbf{u}_{1a}$ is both rotational (its vorticity is mainly opposing the zero order geostrophic vorticity) and divergent. The largest term in the expression of the divergence is

$$\left(\frac{\beta y}{f_0}\right) k_y \sin(k_y y)\widehat{v}_0 \exp\{i(k_x x - \omega_{Ro} t)\}$$

that appears in opposite signs in $\partial u_{1a}/\partial x$ and $\partial v_{1a}/\partial y$, cf. (22). Thus, an estimate of the divergence from

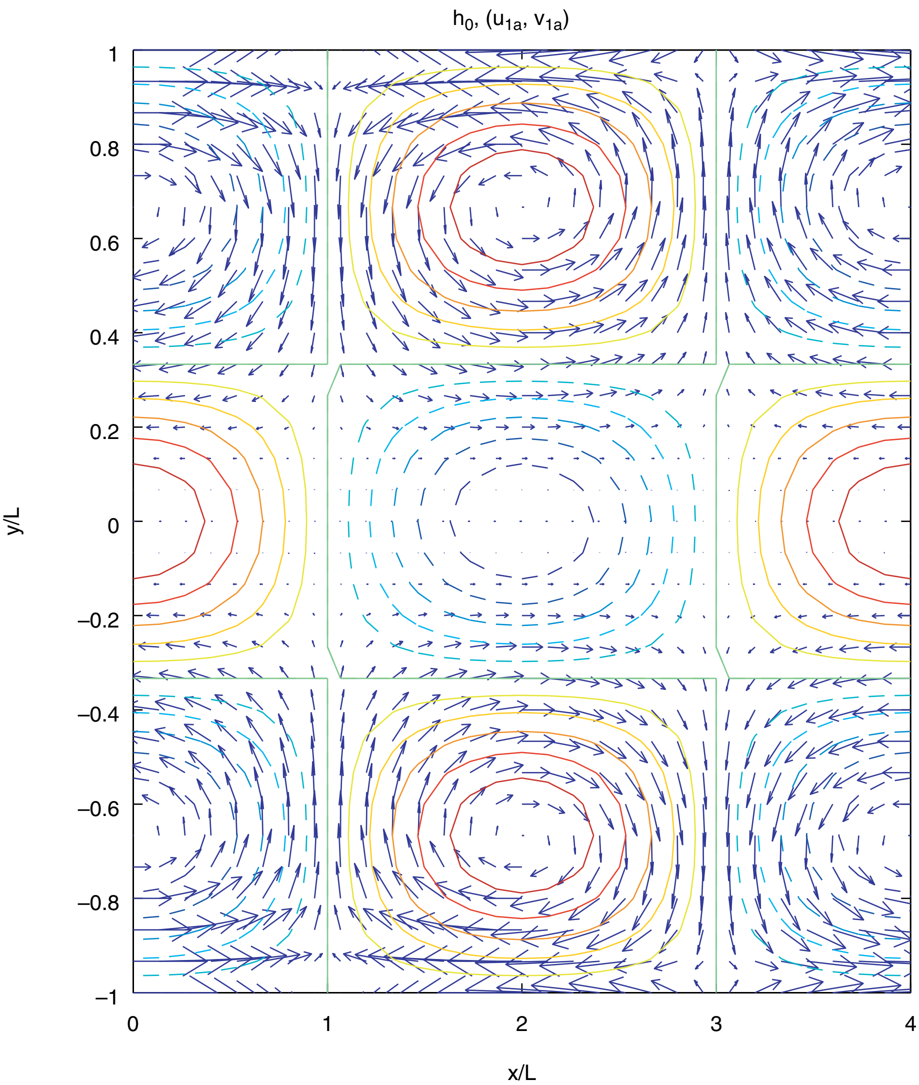


Figure 1. The first-order ageostrophic velocity vector field $\mathbf{u}_{1a}$ (arrows) as obtained from (22), and the zero-order geopotential anomaly $h_0$ as obtained from (4) (with positive and negative values indicated by solid and dashed contours respectively). Here and in Figure 2 we take a channel of meridional width $2L$, where $L = 10^6$ m. Values of the other variables are given in the text. This figure is available in colour online at www.interscience.wiley.com/qj



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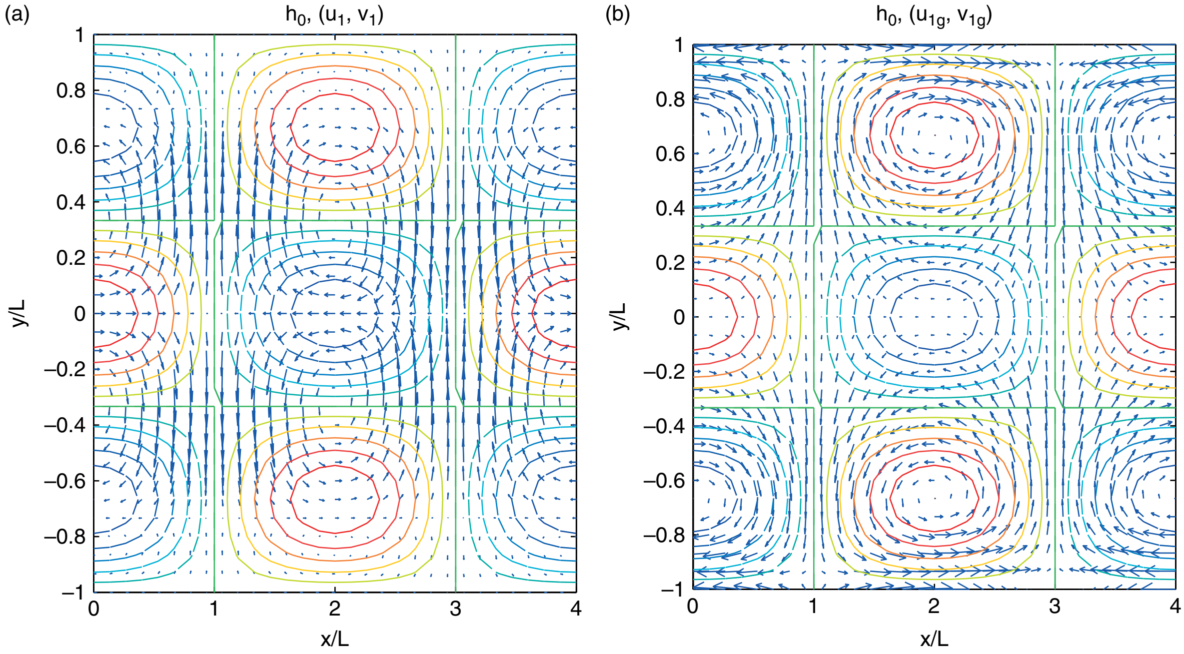


Figure 2. (a) The first-order velocity vector field $\mathbf{u}_1$ (arrows) as obtained from (20a,b), and the zero-order geopotential anomaly $h_0$ as obtained from (4) (with positive and negative values indicated by solid and dashed contours respectively). (b) The first-order geostrophic velocity vector field $\mathbf{u}_{1g} = (g/f_0)\widehat{\mathbf{z}} \times \nabla h_1$ (where $h_1$ is obtained from (20c)), and the zero-order geopotential anomaly $h_0$. This figure is available in colour online at www.interscience.wiley.com/qj

Figure 1 is obtained from a small residual of two large terms.

In Figure 2(a) the total first-order velocity $\mathbf{u}_1$ is plotted versus $h_0$. Note how different $\mathbf{u}_1$ is from $\mathbf{u}_{1a}$ (plotted in Fig. 1) due to the effect of the walls. Since the zero-order velocity of the Rossby wave is geostrophic, where $h_0$ is its stream function (i.e. cyclonic and anticyclonic flows circulate dashed and solid contours, respectively), Figure 2(a) indicates that the first-order correction to the velocity tends to shift the velocity field southwards, as well as the vorticity field anomaly (not shown) and to distort the meridional wavelike structure of the zero-order velocity field. In Figure 2(b) the first-order correction to the geostrophic velocity field, $\mathbf{u}_{1g}$, is plotted versus $h_0$. Since cyclonic and anticyclonic flows of $\mathbf{u}_{1g}$ circulate, respectively, negative and positive geopotential anomaly corrections $h_1$, it is clear from Figure 2(b) that this correction tends to enhance geopotential anomalies in the north and reduce them in the south. A comparison between Figures 1 and 2 reveals that the geostrophic and the ageostrophic components of the correction fields are generally opposing each other, not only at the boundaries but also within the channel interior. As a result, the net first-correction velocity field, $\mathbf{u}_1$, is generally a residue which is smaller than each of the geostrophic and the ageostrophic components of the correcting velocity.

## 3. Concluding remarks

This short note refers to the inconsistency in the concept of 'infinite $\beta$-plane' which underlies the original quasi-geostrophic derivation of Rossby (1939), where on the one hand a finite meridional extension is required ($y \ll f_0/\beta$), while on the other hand the boundaries are set at $y = \pm\infty$.

The analysis presented here provides the $O(\widetilde{\beta})$ correction to the geostrophic velocity for Rossby waves in a zonal channel on the $\beta$-plane. The solution satisfies both the equation and the boundary condition, in contrast to the classical theory on the unbounded $\beta$-plane where the $O(\widetilde{\beta})$ correction describes only the ageostrophic part of the velocity field. The ageostrophic first-order correction to the solution remains unchanged when the infinite $\beta$-plane configuration is bounded by a zonal channel. Since only the ageostrophic component of the first-order eigenfunction is involved in the Rossby wave dynamics, this yields the $O(\widetilde{\beta}^2)$ correction to the wave phase speed to vanish identically, a fact that can actually be anticipated based on the symmetry properties of the governing equations. Hence, the geostrophic first-order component does not take an 'active role' in the first-order dynamics of the Rossby waves (it can be shown that it affects the third-order correction of the phase speed), and in a sense acts to ensure the vanishing of the total meridional velocity component on the boundaries.

The main effect of the boundaries on the Rossby wave is therefore on its meridional structure. The first-order correction distorts the meridional wave-like structure by enhancing the geopotential anomaly on the north and suppressing it on the south of the channel. In contrast, the velocity field, together with the vorticity field, is shifted southwards. The obtained meridional asymmetry (especially the geopotential meridional asymmetry) is in general agreement with the much more sophisticated analysis, performed by Panayotova and Swanson (2006),



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of the edge wave asymmetries obtained in a baroclinic channel. While the latter reveals many other aspects of asymmetry (their analysis was based on the QG+ framework, established by Muraki *et al*. (1999), and included the effect of a zonal jet), our much more modest analysis suggests that a meridional shift of Rossby waves on a $\beta$-plane is inherent in the presence of zonal boundaries and should be taken into account when a zonal channel $\beta$-plane model is being implied.

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