

Apparent Absolute Instability and the Continuous Spectrum

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ABSTRACT

The role of the continuous spectrum and its associated potential vorticity (PV) in absolute instability are investigated in the context of a semi-infinite version of Eady's basic state. This flow crudely resembles the zonally averaged midlatitude atmosphere. The disturbances are composed of a zero PV part and a nonzero PV part. A closed form analytic solution is described that features a localized wave packet whose streamfunction field expands and amplifies in an absolutely unstable way. However, since an examination of the PV field associated with this disturbance reveals that the linear amplification of the localized streamfunction wave packet is induced by a nonlocalized PV field, it is clear that the seed for the expansion of the streamfunction wave packet lies not within the wave packet but upstream of the wave packet. While this precise analytic solution allows for the identification of the upstream PV anomalies, any sort of measurement error would render these upstream PV anomalies invisible. Thus, observations would be incapable of distinguishing an absolute instability seeded by PV anomalies generated within the confines of the streamfunction wave packet from the absolute instability described by the authors' solution.

The general initial value solution is analyzed, and it is found that this apparent absolute instability is not a peculiarity of this particular solution. Absolutely unstable wave packets will be "naturally selected" over geophysically relevant timescales to dominate the flows that emerge from random disturbances to the idealized basic state. In Eady's basic state, which is bounded aloft by a rigid lid, the natural selection mechanism only operates at wavelengths at which the normal modes of Eady's basic state are neutral. It is suggested that an atmospheric counterpart of this natural selection process may be responsible for the medium-scale upper- and lower-tropospheric waves that have recently been identified in the observational record.

The authors prove that the group velocity of a streamfunction field attributable to eastward moving PV anomalies may, in fact, be westward.

1. Introduction

Concerned that normal-mode stability analysis techniques were incapable of explaining localized baroclinic instabilities, Merkin (1977) borrowed mathematical techniques from plasma physics (Briggs 1964) to distinguish *absolute* instability from *convective* instability. According to Briggs (1964, p. 9), a system is absolutely unstable if a disturbance of finite spatial extent grows at every fixed point in space, whereas if the disturbance (eventually) decays at every fixed point in space, the flow is deemed convectively unstable. Although these assessments depend on the frame of reference in which one chooses to define "fixed" points, when one chooses the frame of reference to be fixed relative to the surface

of the earth, they usefully distinguish disturbances that can continuously amplify over a fixed geographical location from those that are advected downstream. Merkin's schematic illustrations of convective and absolute instability are shown in Fig. 1.

Since Merkin's introduction of mathematical techniques for identifying absolutely unstable flows, much of the work on absolute and convective instability has been motivated by the desire to understand the localization and maintenance of midlatitude storm tracks. As was noted by Blackmon (1976) and Lau and Wallace (1979), Northern Hemisphere baroclinic activity tends to be concentrated in zonally confined storm tracks. While Hoskins and Valdes (1990) argued, among other things, that since the storm tracks are located downstream of, and at, regions where small disturbances are likely to rapidly amplify and since there are sources of disturbances upstream of storm tracks, the position of the storm tracks can be accounted for by waves that are merely convectively unstable. On the other hand, in the

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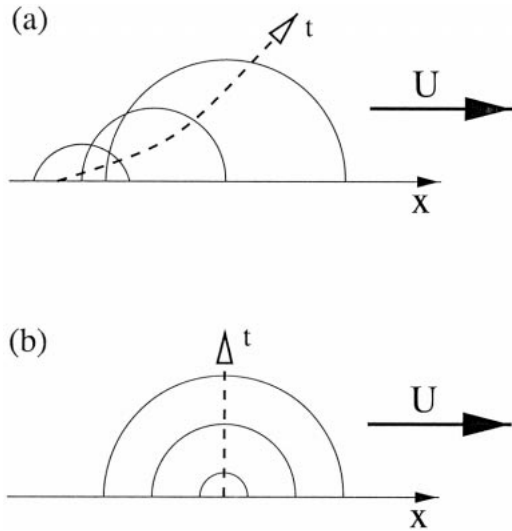


FIG. 1. Schematics of convective and absolute instability development are shown in (a) and (b), respectively. In convective instabilities, the perturbation (illustrated by the round curves) is advected downstream as it evolves with time t (illustrated by the dashed arrow). In absolute instabilities, the perturbation resists the mean flow advection and remains stationary as it amplifies and expands with time.

context of WKBJ theory, Pierrehumbert's (1984) work has shown that the existence of localized regions of absolute instability allows localized eigenmodes in certain circumstances. The theory produces similarities to observed storm track structure when applied to inhomogeneous baroclinic jets. In this way, Pierrehumbert's theory suggests that while the existence of absolute instability is not a necessary condition for the existence of storm tracks, it may be a sufficient condition. Swanson and Pierrehumbert (1995) have gone on to point out that, *inter alia*, the existence of absolute instability allows for the possibility of two apparent sources of excitation—one upstream and one within the storm track.

The ability of baroclinic wave packets to resist being advected downstream is also relevant to theories for the development of waves on atmospheric fronts.¹ Since atmospheric fronts have a finite length, amplifying packets of frontal waves may be advected off the front before they reach a significant amplitude. If a wave packet can resist the tendency of the wind to advect disturbance energy downstream, it has more of an opportunity to produce a significant frontal wave than it would if it was rapidly blown off the frontal region.

Here, we describe how a localized, absolutely unstable, baroclinic streamfunction disturbance may be seeded by disturbances in the potential vorticity (PV) field far upstream of the disturbance. Importantly, the upstream PV anomalies make a vanishingly small impact

on the amplitude of the streamfunction field until they are advected into the center of the disturbance. Once they are in the wave packet, they have a large impact on the streamfunction and cause surface temperature anomalies to amplify. Implications of this demonstration include the following. 1) Absolute instability of the streamfunction field does not necessarily imply the existence of a source of excitation capable of moving upstream from within the wave packet to beyond the leading edge of the wave packet, and 2) the accurate representation of the absolute instability described by our solution requires higher spatial resolution than that provided by discrete observation–modeling–forecasting systems.

Our interpretation of the general initial value solution for the semi-infinite Eady model expands and generalizes previous analytic work on the semi-infinite Eady problem by Farrell (1984), Müller et al. (1989), Thorncroft and Hoskins (1990), Chang (1992), and Davies and Bishop (1994). Farrell gave a particular solution relevant to midlatitude cyclogenesis that emphasized the importance of the continuous spectrum in development. Müller et al. (1989) gave a zero PV perturbation solution relevant to both frontogenesis and cyclogenesis that did not include the continuous spectrum. Thorncroft and Hoskins presented an analytic initial-value solution relevant to frontal wave development. Chang (1992) clarified the connection between Thorncroft and Hoskins's work and Farrell's work and also showed how the semi-infinite Eady model could be used to understand how vertical discretization in numerical models could lead to forecast errors. Davies and Bishop (1994) provided an interpretation of development in the semi-infinite Eady model in terms of the interaction between interior PV waves and potential temperature waves on the surface.

In section 2, we present a closed-form initial value solution that features a localized absolutely unstable streamfunction disturbance. To show that these disturbances arise naturally from random perturbations to the semi-infinite Eady basic state in section 3, we derive the general initial value solution and show how it may be decomposed into wave packets of distinct group velocities. In doing so, we explicitly isolate the absolutely unstable part of the solution. Our conclusions are drawn in section 4.

2. Dynamics of an absolutely unstable wave packet

Our example of an absolutely unstable wave packet is obtained by letting PV waves of the type discussed in Thorncroft and Hoskins (1990) resonantly amplify the zero PV nonamplifying wave packet discussed in Müller et al. (1989). We begin by describing the basic state and the equations governing the evolution of perturbations to the basic state. We then review Müller et al.'s (1989) neutral wave packet and Davies and Bishop's (1994) interpretation of Thorncroft and Hoskins's

¹ Parker (1998) provides a review of current theories of frontal wave development.

(1990) PV waves. Finally, we select the set of PV waves that amplifies Müller et al.'s (1989) neutral wave packet.

a. The basic state

The semi-infinite Eady basic state considered here would be equivalent to Eady's (1949) model were it not for the fact that in the semi-infinite Eady model there is no upper boundary. This semi-infinite basic state comprises a quasigeostrophic, incompressible, Boussinesq, and uniformly stratified flow on an f plane in the space $z \in [0, \infty)$. The basic state has a linear uniform baroclinic shear $\bar{U} = \Lambda z$ and hence a zero mean PV gradient. The PV equation for a y -independent PV disturbance is

$$\left(\frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x}\right) \left(\psi'_{xx} + \left(\frac{f_0}{N}\right)^2 \psi'_{zz}\right) = 0, \quad (1)$$

where ψ' is the geostrophic perturbation streamfunction, f_0 is the domain-averaged Coriolis parameter, and N is the Brunt-Väisälä frequency. A simplifying coordinate transformation for the perturbation equations for this basic state is

$$(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \left(x, y, \frac{N}{f_0} z, \frac{f_0 \Lambda}{N} t\right). \quad (2)$$

Note that although \tilde{z} is stretched relative to z , it is measured in fractions of meters while \tilde{t} is nondimensional. We chose not to nondimensionalize spatial coordinates because there is no obvious limited spatial scale associated with the basic state. In these transformed coordinates, the PV equation (1) takes the form

$$\left(\frac{\partial}{\partial \tilde{t}} + \tilde{z} \frac{\partial}{\partial \tilde{x}}\right) (\psi'_{\tilde{x}\tilde{x}} + \psi'_{\tilde{z}\tilde{z}}) = 0, \quad (3a)$$

For reference, note that for the geophysically relevant midlatitude parameter values are $\Lambda = 30 \text{ m s}^{-1} (10 \text{ km})^{-1}$, $f_0 = 10^{-4} \text{ s}^{-1}$, and $N = 10^{-2} \text{ s}^{-1}$; then units of \tilde{z} and \tilde{t} would correspond to 0.01 m and 9.26 h, respectively. The surface boundary condition for (3a) is obtained from the thermodynamic equation together with the requirement that vertical velocity vanishes at $\tilde{z} = 0$; that is,

$$\left(\frac{\partial}{\partial \tilde{t}} + \tilde{z} \frac{\partial}{\partial \tilde{x}}\right) \psi'_{\tilde{z}} = \psi'_{\tilde{x}}. \quad (3b)$$

Note that $\psi_{\tilde{z}}$ and $\psi_{\tilde{x}}$ are scaled potential temperature θ' and geostrophic meridional wind perturbations v'_g , respectively. The second boundary condition for (3a) comes from the requirement that ψ' remain finite as $\tilde{z} \rightarrow +\infty$. For notational convenience, for the rest of this paper we shall drop the tildes and let (x, y, z, t) refer to the coordinates $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ defined in Eq. (2).

b. Müller et al.'s zonally localized neutral wave packet

Müller et al. (1989) took the Fourier integral of the y -independent form of the discrete zero PV neutral-mode solutions discussed in Gill (1982, section 13.2),

$$\psi_\theta = \int_{k=-\infty}^{\infty} -\frac{\theta_0(k)}{\mu} e^{-\mu z} e^{ik[x-(t/\mu)]} dk, \quad (4)$$

where $\mu = \sqrt{k^2} = |k|$, with $\theta_0(k) = -\mu A e^{-b\mu}/2$, to obtain

$$\psi_\theta = \frac{1}{2} \int_{k=-\infty}^{\infty} A e^{-b\mu} e^{-\mu z} e^{i[kx-(k/\mu)t]} dk. \quad (5a)$$

Note that the group velocities of the waves compose this integral are precisely equal to zero. Grouping together the parts of the integrand into odd and even parts allows the above expression to be rewritten in the form

$$\begin{aligned} \psi_\theta &= \cos(t) \int_{k=0}^{\infty} A e^{-k(b+z)} \cos(kx) dk \\ &+ \sin(t) \int_{k=0}^{\infty} A e^{-k(b+z)} \sin(kx) dk. \end{aligned} \quad (5b)$$

Using the appropriate standard integral formula, one then obtains the Müller et al. (1989) solution

$$\psi_\theta = A \cos(t) \frac{\alpha_1}{x^2 + \alpha_1^2} + A \sin(t) \frac{x}{x^2 + \alpha_1^2}, \quad (5c)$$

where $\alpha_1 = b + z$. As discussed in Hoskins et al. (1985) and Bishop and Thorpe (1994), this wave packet may be viewed as being attributable to the potential temperature distribution at the surface.

The wave packet oscillates with the period 2π that, with the typical parameter values mentioned in section 2a, corresponds to a period of 58.18 h. As Müller et al. point out, even though the envelope of this wave packet does not grow, cyclonic vorticity increases as the initial surface anticyclone, $\psi_\theta(x, z = 0, t = 0) = A(b/\chi^2 + b^2)$, is replaced by a cyclone in 29.09 h. In the next section, we show how the envelope of this wave packet can be amplified by PV waves in the interior of the domain.

c. Resonance-inducing waves of PV

In the appendix of Thorncroft and Hoskins (1990) it is noted that when an infinitely thin PV wave of zonal wavenumber k is advected at the same speed as the propagation speed of the zero PV neutral mode described by (4), it (eventually) causes a zero PV neutral mode to amplify linearly with time. This linear amplification rate is proportional to the magnitude of the PV wave. They also show that in order for the PV wave to be advected at the same speed as the propagation speed of its zero PV counterpart, its infinitely thin PV field must be placed at the Rossby height, $H_R = 1/\mu$. The relevance of infinitely thin PV waves to development

in our semi-infinite basic state is illuminated by the Green's function representation of the evolution of an arbitrary PV disturbance $PV(x, z, t)$; namely,

$$PV(x, z, t) = \int_{k=-\infty}^{\infty} \int_{z'=0}^{\infty} P(z', k) \delta(z' - z) e^{ik(x-z't)} dz' dk. \tag{6a}$$

Here, the prime on z is used to distinguish the height z' over which the integral is being performed from the

(stretched) height z , and $P(z, k)$ is the Fourier transform in the x direction of the PV field at the height z' . This equation gives the evolution of the PV field in terms of an integral of infinitely thin PV waves of wavenumber k moving at the speed z' of the local wind. Note that in the scaled coordinates $\tilde{z}'\tilde{t} = \overline{U}(z')t$.

As discussed in Davies and Bishop (1994), the streamfunction field whose PV field is identical to that given by (6a) and whose surface potential temperature distribution is equal to zero is given by

$$\psi_{IPV}(x, z, t) = \int_{k=-\infty}^{\infty} \int_{z'=0}^{\infty} - \left[P(z', k) \frac{\gamma(z', z)}{\mu} e^{ik(x-z't)} \right] dz' dk, \tag{6b}$$

where

$$\gamma(z', z) = [1 + \tanh(\mu z')]^{-1} \begin{cases} \cosh(\mu z)/\cosh(\mu z') & \text{for } z < z' \\ e^{-\mu(z-z')} & \text{for } z > z' \end{cases}$$

(the subscript IPV stands for internal PV). Again, following Bishop and Thorpe (1994) and Hoskins et al. (1985), since this streamfunction field has zero surface potential temperature associated with it, it may be viewed as being solely attributable to the internal PV field described by (6a). The $[1 + \tanh(\mu z')]$ term is necessary to ensure that $\gamma(z', z'+) - \gamma(z', z'-) = [(1 + \tanh(\mu z'))^{-1} [-\mu - \mu \tanh(\mu z')]] = -\mu$. This relation ensures that the amplitude of the x -Fourier transform of the PV field at the height z' is proportional to $P(z', k)$.

Comparing Eqs. (6a) and (6b), we see that the streamfunction field attributable to an infinitely thin PV wave located at the height z' with wavenumber k is given by the negative of the term in square brackets in Eq. (6b). The specific structure of this streamfunction field is illustrated in Fig. 2. Note that the wind field attributable to each infinitely thin PV wave is nonzero at the surface. Since the meridional component of this wind advects the surface mean temperature, any arbitrary distribution of interior PV results in a redistribution of the surface potential temperature field. The streamfunction field associated with these changes in surface potential temperature does not change the interior PV field because the variation of the PV field with y is assumed to be zero. (Note that even if the PV field did vary with y the advection of it by winds attributable to surface potential temperature would be a nonlinear process and hence negligible for small amplitude disturbances.)

The streamfunction field attributable to surface potential temperature has zero PV associated with it. Hence, it must take the zero PV form,

$$\psi_{\theta}(x, z, t) = \int_{k=-\infty}^{\infty} \int_{z'=0}^{\infty} - \frac{B(k, z', t)}{\mu} e^{-\mu z} e^{ikx} dz' dk. \tag{7a}$$

The corresponding potential temperature field is

$$\theta(x, z, t) = \psi_z(x, z, t) = \int_{k=-\infty}^{\infty} \int_{z'=0}^{\infty} B(k, z', t) e^{-\mu z} e^{ikx} dz' dk, \tag{7b}$$

where $B(k, z', t)$ denotes a complex amplitude of the component of the zero PV surface wave excited by PV at the height z' with wavenumber k ($B = 0$ at $t = 0$ since we assume zero surface potential temperature disturbance at $t = 0$). In mathematical terms, the interaction between the wind field attributable to interior PV and the surface temperature field is described by using (6b) and (7a,b) in (3b) to obtain

$$\frac{\partial B}{\partial t} = \frac{-ik}{\mu} [B + P(z', k) e^{-\mu z'} e^{-ikz't}]. \tag{8}$$

The solution to this differential equation is

$$B = \frac{e^{-i(k/\mu)t} P(z', k) e^{-\mu z'}}{\left(\frac{1}{\mu} - z'\right)} [1 - e^{ik[(1/\mu)-z']t}]. \tag{9a}$$

To see the behavior of this solution near $z' = 1/\mu$, multiply both the numerator and denominator of this equation by $e^{i(k/2)(1/\mu+z')t}$ and make use of the Euler formula to obtain

$$B(k, z', t) = \frac{-iP(z', k) e^{-\mu z'}}{\frac{\mu}{2} \left(\frac{1}{\mu} - z'\right)} \sin \left[\frac{k}{2} \left(\frac{1}{\mu} - z'\right) t \right] \times e^{-i(k/2)(1/\mu+z')t}. \tag{9b}$$

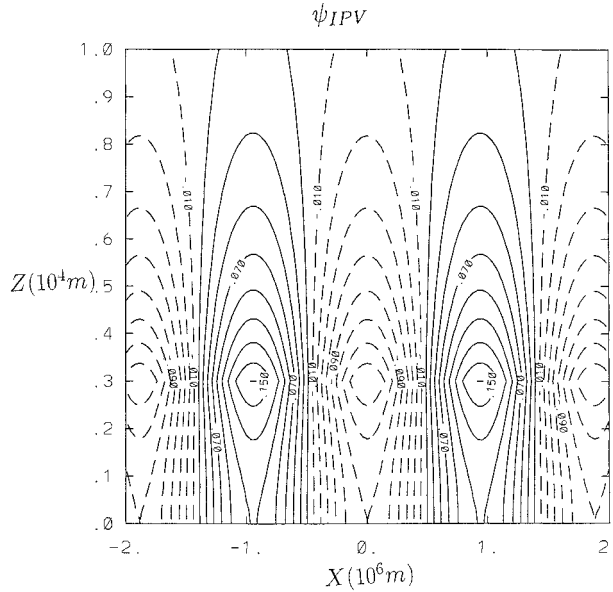


FIG. 2. A zonal vertical cross section of the streamfunction ψ_{IPV} , attributable to a PV kernel located at 3 km. The associated resonant wavelength for the dimensional parameters given in section 2a is $\cong 1900$ km. Note that the field attributable to this PV kernel has zero surface potential temperature. The units of the streamfunction are normalized here by $P(z', k) \times 10^6$, where z' and k pertain to 3 km and $2\pi/1900$ km, respectively. Solid and dashed lines indicate positive and negative values, respectively.

When the PV wave is located exactly at the Rossby height of deformation, $z' = 1/\mu$, its dimensional period $T = 2\pi N/f_0 \Lambda$ is the same as that of the surface modes, and resonant surface-mode amplification ensues. To see this in the mathematics, take the limit of (9b) as z' tends to $1/\mu$ to obtain

$$B\left(k, \frac{1}{\mu}, t\right) = \left[-i \frac{P\left(\frac{1}{\mu}, k\right)}{e} \frac{k}{\mu} e^{-i(k/\mu)t} \right] t. \quad (10)$$

Since the magnitude of P is constant, it is clear from (10) that B undergoes a linear amplification. As discussed in Davies and Bishop (1994), this growth results from the fact that the PV crests lag surface potential temperature crests by $\pi/2$ as both crests move to the east at the speed $1/\mu$.

d. Absolute instability via deformation of PV

Imagine that there was a single infinitely thin PV wave at every possible height z' of the atmosphere and that the wavenumber of each PV wave was the resonant wavenumber, $\mu_r = 1/z'$. In this case,

$$P(z', k) = F(z') \delta\left(z' - \frac{1}{\mu}\right); \quad (11)$$

that is, the Fourier transform of the PV at the height z' has δ -function peaks at $k = \pm 1/\mu$. Note that this does

not imply that the PV is singular—it merely implies that the PV field at z' is entirely described by a finite-amplitude wave with wavelength $\lambda = 2\pi/\mu$. Using (11) and (10) in (7a) gives

$$\psi_\theta(x, z, t) = \left[\frac{i}{e} \int_{-\infty}^{\infty} \frac{F\left(\frac{1}{\mu}\right)}{\mu} e^{-\mu z} \frac{k}{\mu} e^{-i(k/\mu)t} e^{ikx} dk \right]. \quad (12a)$$

Since $F(1/\mu)$ is symmetric about $k = 0$, and since k/μ is equal to the sign of k , we may group the odd and even parts of the integral in (12a) to obtain

$$\psi_\theta(x, z, t) = \frac{2}{e} \left[\sin(t) \int_0^{\infty} \frac{F\left(\frac{1}{\mu}\right)}{\mu} e^{-\mu z} \cos(kx) dk - \cos(t) \int_0^{\infty} \frac{F\left(\frac{1}{\mu}\right)}{\mu} e^{-\mu z} \sin(kx) dk \right] t. \quad (12b)$$

Similarly, using (11) in (6b) gives

$$\psi_{IPV}(x, z, t) = -2 \left[\cos(t) \int_0^{\infty} \frac{F\left(\frac{1}{\mu}\right) \gamma\left(\frac{1}{\mu}, z\right)}{\mu} \cos(kx) dk + \sin(t) \int_0^{\infty} \frac{F\left(\frac{1}{\mu}\right) \gamma\left(\frac{1}{\mu}, z\right)}{\mu} \sin(kx) dk \right]. \quad (13)$$

In this way, we see that the class of PV perturbations defined by (6b) subject to (11) leads to a linear amplification of the zero PV streamfunction field (12b). This amplification is caused by the resonant interaction described Thorncroft and Hoskins (1990). In the terminology of Pedlosky (1964), the surface modes grow as a result of their interaction with the continuous spectrum of modes required to represent the interior PV field.

In order to highlight the implications of this continuous spectrum on absolute instability, we need to choose $1/\mu$ such that a zonally localized disturbance results. Muller et al.'s (1989) solution suggests an analytically tractable localized disturbance. Comparing (12) with the kernel used in (5), we see that Muller et al.'s wave packet is excited if we take

$$F\left(\frac{1}{\mu}\right) = -\frac{A\mu e^{(1-b\mu)}}{2}. \quad (14)$$

As is shown in appendix A, substituting this choice for F in (13) gives

$$\begin{aligned} \psi_{\text{IPV}}(x, z, t) = \frac{A}{2} & \left\{ \cos(t) \left[\frac{\alpha_1}{\alpha_1^2 + x^2} + \frac{\alpha_2}{\alpha_2^2 + x^2} + \frac{e^{-(\alpha_2/z)}}{\alpha_2^2 + x^2} \left[x \sin\left(\frac{x}{z}\right) - \alpha_2 \cos\left(\frac{x}{z}\right) \right] + \frac{e^{[2-(\alpha_1/z)]}}{\alpha_1^2 + x^2} \left[\alpha_1 \cos\left(\frac{x}{z}\right) - x \sin\left(\frac{x}{z}\right) \right] \right] \right. \\ & + \sin(t) \left[\frac{x}{\alpha_1^2 + x^2} + \frac{x}{\alpha_2^2 + x^2} - \frac{e^{-(\alpha_2/z)}}{\alpha_2^2 + x^2} \left[\alpha_2 \sin\left(\frac{x}{z}\right) + x \cos\left(\frac{x}{z}\right) \right] \right. \\ & \left. \left. + \frac{e^{[2-(\alpha_1/z)]}}{\alpha_1^2 + x^2} \left[x \cos\left(\frac{x}{z}\right) + \alpha_1 \sin\left(\frac{x}{z}\right) \right] \right] \right\}, \end{aligned} \quad (15)$$

where $\alpha_1 = b + z$ and $\alpha_2 = b - z$. Although this equation is singular for certain values of x and z , the limit analysis discussed in appendix B shows that ψ_{IPV} is bounded throughout the semi-infinite domain. The streamfunction field attributable to the surface potential temperature field is obtained by substituting (14) into (12b) to obtain

$$\psi_\theta = A \left[\cos(t) \frac{x}{\alpha_1^2 + x^2} - \sin(t) \frac{\alpha_1}{\alpha_1^2 + x^2} \right] t. \quad (16)$$

Equation (5b) describes a zonally localized wave packet that differs from the neutral wave packet (5c) only in that 1) the oscillation of the wave packet is a $\frac{1}{4}$ period ahead of that in (5c) and 2) it is linearly amplifying with time. This linear amplification occurs at all values of x ; hence, the evolution of the wave packet is in accord with Merkin's schematic picture (Fig. 1b) of absolute instability.

More important, it is a zonally localized region of amplifying baroclinic activity whose center is stationary even though the wind is westerly everywhere above the surface. This fact is illustrated in Figs. 3, 4, and 5, which show the evolution of the total streamfunction, $\psi' = \psi_{\text{IPV}} + \psi_\theta$ and its component parts. Figure 5, which shows the streamfunction field attributable to the interior PV field, is particularly interesting. It shows streamfunction anomalies apparently appearing out of nothing in the air upstream of the wave packet center. The process by which this occurs can be inferred from the interior PV field,

$$\text{PV} = -\frac{A}{z^3} e^{[1-(b/z)]} \left[\cos(t) \cos\left(\frac{x}{z}\right) + \sin(t) \sin\left(\frac{x}{z}\right) \right], \quad (17)$$

which can be obtained either from the Laplacian of (15) or, as is shown in appendix C, by using (11) and (14) in (6a). The evolution of this field is illustrated in Fig. 6. This figure shows that the apparently quiescent upstream and downstream regions are packed with waferlike PV perturbations. In these regions, positive and negative PV wafers overlie each other rather like the pages of a book. The close proximity and fine structure of these positive and negative PV wafers in the upstream region result in the streamfunction field attributable to the positive PV wafers canceling out the streamfunction field attributable to the negative PV wafers. In time, thermal wind shear transforms the PV wafers from hor-

izontally lying thin PV anomalies into vertically standing broad PV anomalies. As the anomalies are advected into the vertical position, their distance from oppositely signed regions of PV is maximized and consequently, their impact on the streamfunction field is maximized.

The surface wind field attributable to these broad PV anomalies causes the surface temperature field to amplify at a relatively rapid rate at the center of the wave packet. Eventually, the thermal wind shears the PV anomalies back down into a nearly horizontal position downstream of the wave packet where they have a vanishingly small impact on the streamfunction field.

Note that the PV field is neither localized nor amplifying. The PV field does not satisfy the requirements for absolute instability, but the streamfunction field it induces does. Since one of the fields associated with our streamfunction field is not absolutely unstable, one could argue that our amplifying wave packet is not "truly" absolutely unstable. However, if a disturbance of the type described by (15) and (17) occurred in a laboratory (or atmospheric) environment, the imprecision of measuring instruments would prevent the upstream PV filaments from being resolved. Hence, in a laboratory (or atmospheric) environment, it would appear that the seed of the absolute instability lay within the wave packet itself; that is, it would appear that the disturbance was truly absolutely unstable.

Another related reason why this type of instability only classifies as an apparent absolute instability and not a true absolute instability is that our initial disturbance is vanishingly small but non-zero at $|x| \rightarrow \infty$. Analysis, not entered into here, shows that if the initial streamfunction field were localized and precisely equal to zero at an upstream of a certain location, it would remain equal to zero upstream of this location. However, as mentioned above, in a laboratory (or atmospheric) environment it would be impossible to distinguish the nonlocalized disturbance described by (15) and (16) from a disturbance of finite spatial extent. Consequently, the growth of our disturbance at all x values might be mistaken for the upstream movement of initially localized energy.

Since the above is a *particular* solution to the governing equations, one must ask whether similarly absolutely unstable structures would result from random disturbances to the basic state.

3. The natural selection of absolutely unstable wave packets from random disturbances

a. Sorting wave packets by their group velocity

We begin by describing the initial-value solution for arbitrary disturbances to our semi-infinite basic state.

$$\psi_{\theta}(x, z, t) = \int_{k=-\infty}^{\infty} -\frac{1}{\mu} \left[\theta_0(k) e^{-t(k/\mu)t} + \int_0^{\infty} B(k, z', t) dz' \right] e^{-\mu z} e^{ikx} dk. \quad (18)$$

[Recall that $B(k, z', 0) = 0$.] Note that the addition of surface potential temperature at $t = 0$ does not change the way in which interior PV evolves because there is no interior PV gradient and the disturbances have no y dependence. Even if the disturbances did have y dependence, the interaction between the wind field attributable to surface potential temperature and interior PV anomalies would be negligible while wave amplitudes remained small enough for linear dynamics to dominate. The addition of ψ_{IPV} (6b) and ψ_{θ} (18) gives the initial value solution for the evolution of any arbitrary streamfunction disturbance to the semi-infinite basic state that is evanescent as z tends to infinity.

We now seek to rewrite this solution in a form that 1) places the amplifying solutions associated with PV fields of the form given by (11) in a more general context, and 2) demonstrates how it is possible, in the absence of an internal basic state PV gradient, for the group velocity of a streamfunction field to be westward even when the PV it is associated with is blowing eastward. The lack of interior PV gradients in our basic-state flow mean that the relationships between group velocity and eddy correlations that have been discussed in the context of \mathbf{E} vectors, Eliassen–Palm fluxes, and wave activity, for example, James (1994), are not applicable to our basic state. Such relationships only apply to PV perturbations that result from displacements of the basic-state PV field that conserve PV. While the perturbation PV in our basic state cannot be produced by simply displacing the basic-state PV, it could be produced by processes such as latent heating/cooling, radiative heating/cooling, and internal friction, all of which do not conserve PV. These classes of disturbances are not accounted for by the aforementioned relationships between eddy correlations and group velocities.

To understand how a wide range of group velocities

Equations (6), (7), and (9) give the initial-value solution for any disturbance that has zero surface potential temperature at $t = 0$. To embrace the possibility of nonzero potential temperature at $t = 0$, we simply need to add to (7a), the zero PV initial value solution described by Eq. (4). The resulting expression is

can be achieved by the streamfunction wave packets in our idealized basic state, recall from (6b) that the frequency of the streamfunction field attributable to infinitely thin PV waves at the height z' is equal to the speed of the wind at z' multiplied by the wavenumber, k , of the wave. The group velocity of the streamfunction field attributable to two PV waves of slightly differing wavenumbers centered at slightly different heights is determined by the ratio of the differences in the frequencies to the differences in wavenumbers. For example if, as was the case in section 2, the difference between the frequencies at the two heights is zero, then the group velocity of the wave packet associated with the streamfunction field attributable to the two PV waves will also be zero. More generally, by letting

$$z' = \frac{1}{\mu} + \eta, \quad (19)$$

where η is a scalar variable greater than $-1/\mu$, the frequency of the streamfunction field attributable to the interior PV is given by

$$\omega = kz' = \frac{k}{\mu} + k\eta. \quad (20)$$

For $k \neq 0$, the group speed along the x axis is given by

$$c_g = \frac{\partial \omega}{\partial k} = \eta. \quad (21)$$

Thus, by holding η fixed and letting the wavenumber of the PV waves vary with height according to (19), one obtains a streamfunction wave packet with a group velocity of η . This idea is formalized in appendix D where we show that the initial value solution for the part of the streamfunction field given by (6b) can be rewritten in the form

$$\psi_{\text{IPV}}(x, z, t) = \int_{\eta=-\infty}^{\infty} \int_{k=-\infty}^{\infty} -\mathcal{H}\left(\eta + \frac{1}{\mu}\right) \frac{\gamma[z'(\eta, k), z]}{\mu} P[z'(\eta, k), k] e^{-i(k/\mu)t} e^{ik(x-\eta t)} dk d\eta, \quad (22)$$

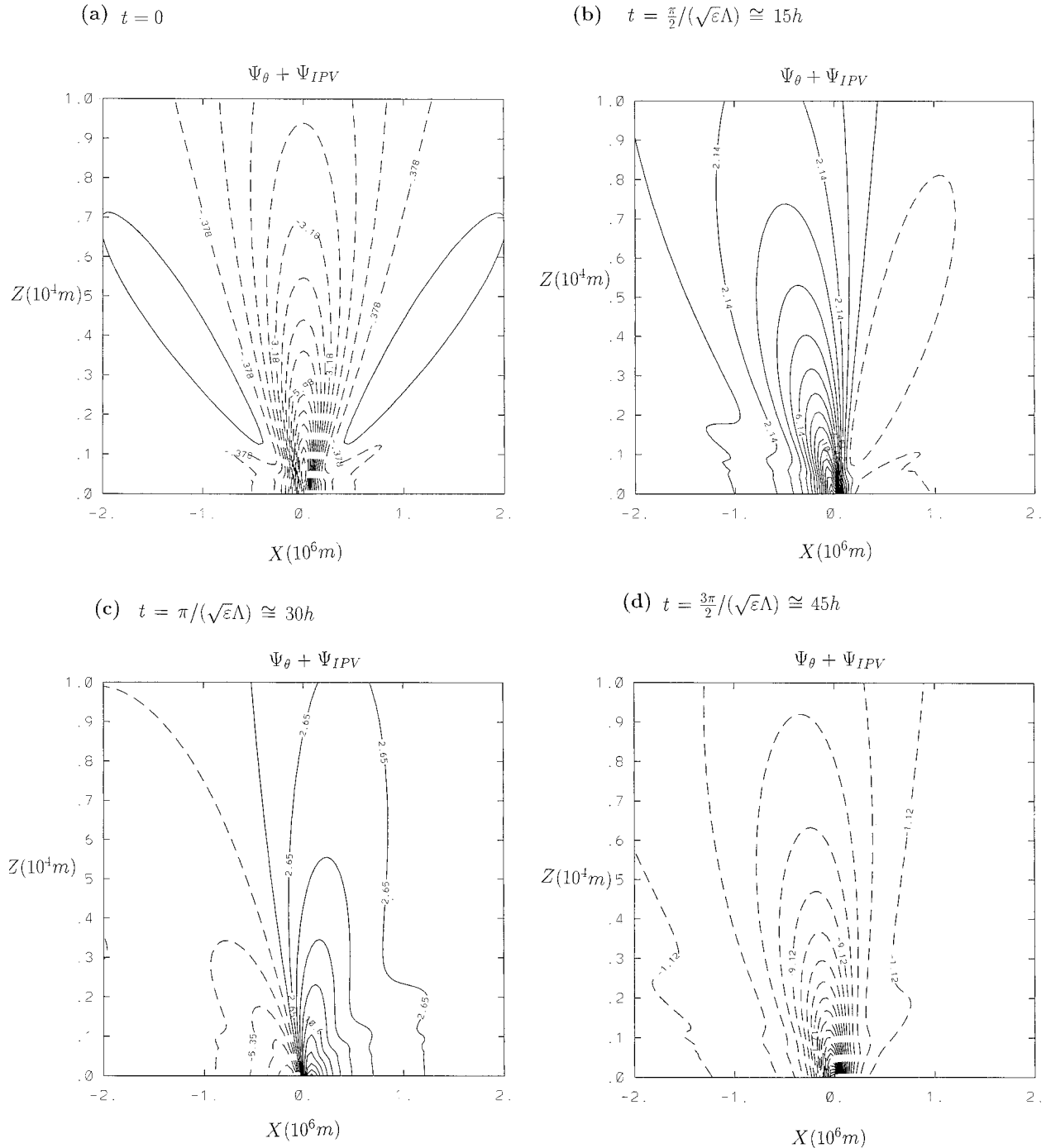


FIG. 3. Zonal vertical cross sections of the sum of the internal PV field streamfunction ψ_{IPV} and the surface temperature streamfunction ψ_θ , as described in Eqs. (15) and (16), respectively. The cross sections are plotted for times (a) $t = 0$, (b) $t = (\pi/2)/(f\Lambda/N) \cong 15$ h, (c) $t = \pi/(f/N)\Lambda \cong 30$ h, (d) $t = (3\pi/2)/(f/N)\Lambda \cong 45$ h. The units of the streamfunction are normalized here by $A \times 10^{-6}$, where solid and dashed lines indicate positive and negative values, respectively.

where \mathcal{H} is the unit step function. This equation gives ψ_{IPV} in terms of an integral over group velocities of wave packets of distinct group velocities η . Note that the oscillation frequency of all of these modes is determined

by the $e^{-i(k/\mu)t} = e^{-i\text{sign}(k)t}$ term. As this term is independent of η , all of the wave packets have the same period of oscillation, $T = 2\pi$ (or 58.18 h, for the typical atmospheric parameters mentioned in section 2a).

In appendix E, the critical dependence of this oscillation period on how one chooses to group together the continuous-spectrum streamfunction waves is illustrated by demonstrating how the substitution $z' = \xi/\mu$ groups continuous-spectrum modes into wave packets with dif-

fering oscillation periods but equivalent group velocities ($c_g = 0$ for all wave packets).

To use (22) to isolate a wave packet having the constant group velocity η_c , one simply sets $P[z'(\eta, k), k] = G(k)\delta(\eta - \eta_c)$ in (22), where $G(k)$ is an arbitrary function of k . With this choice (22) reduces to

$$\psi_{\text{IPV}}(x, z, t) = \left\{ \int_{k=-\infty}^{\infty} -\mathcal{H}\left(\eta_c + \frac{1}{\mu}\right) \frac{\gamma[z'(\eta_c, k), z]}{\mu} G(k) e^{ik(x-\eta_c t)} dk \right\}. \quad (23)$$

The unit step function \mathcal{H} in this expression dictates that while wave packets with positive (eastward) group velocities may contain waves with wavenumbers of all magnitudes, wave packets with negative (westward) group velocities can only contain waves whose wavenumber magnitudes are less than $\Lambda(f/N)/\eta_c$. Nevertheless, Eqs. (22) and (23) clearly indicate that wave packets of *any* group velocity are possible. Thus, eastward PV advection does not necessarily imply that its corresponding streamfunction field moves eastward—on the contrary, the field can

propagate westward. This fact does not appear to be widely appreciated.

b. The natural selection of zero group velocity wave packets over geophysically relevant time- and space scales

To see how the response of the surface temperature field depends on the group velocity of the PV wave packet that induces the response, we 1) set $\theta_0 = 0$ in (18); 2) use (19) in (9b) to obtain an expression for B in terms of k , η , and t ; and 3) use $B(k, \eta, t)$ in (18) to obtain

$$\psi_\theta(x, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i\mathcal{H}\left(\eta + \frac{1}{\mu}\right) \left[\frac{e^{-(1+\eta\mu)P}}{\mu} \right] \frac{\sin\left(\frac{k\eta t}{2}\right)}{\frac{\mu\eta}{2}} e^{-\mu z} e^{-i(k/\mu)t} e^{ik[x-(\eta/2)t]} dk d\eta. \quad (24)$$

Consequently, waves for which $k\eta t/2$ is small approximately satisfy the relation

$$\sin(k\eta t/2) = k\eta t/2 \quad (25)$$

and hence experience linear growth. For a given η value, the amount of time a wave can grow linearly depends on its wavenumber k . Thus, in considering what the above equation can tell us about the types of disturbances random PV distributions are likely to excite in earth's atmosphere, we must first identify the range of wavenumbers over which our solutions are relevant to the atmosphere.

First, in order to avoid violating the assumptions of quasi-geostrophic theory, we must exclude very high wavenumber modes from our consideration. We choose to make a generous estimate of the range of wavenumbers over which quasigeostrophic theory has predictive power and exclude only those wavenumbers for which $|k| > 2 \times 10^{-5} \text{ m}^{-1}$ (i.e., only waves with wavelengths greater than $100\pi \text{ km}$ are included). Assuming

a velocity scale U_s of 10 m s^{-1} , this gives an upper limit to the Rossby number of $\text{Ro} = U/f_0 L = 1/\pi$.

Second, since the terrestrial atmosphere includes a tropopause at around $H = 10 \text{ km}$, we should also exclude waves that have a significant interaction with the tropopause. Eady (1949) showed that the addition of a rigid lid at a height H to the basic state being considered here leads to a dispersion relation for lower-level neutral modes of the form

$$\omega_{\text{Eady}} = \frac{\tilde{H}}{2} k \left[1 - \sqrt{1 - \frac{4}{\mu\tilde{H} \tanh(\mu\tilde{H})} + \left(\frac{2}{\mu\tilde{H}}\right)^2} \right]. \quad (26)$$

(cf., e.g., Holton 1992, p. 260), where $\tilde{H} = NH/f_0$ is the Rossby radius of deformation and where we have assumed the coordinate transformation given in section 2a. Note that multiplying the right-hand side of this equation by the inverse timescale $f_0\Lambda/N$ recovers the exact form given in Holton. From (4), the dispersion relation for the semi-infinite basic state is just

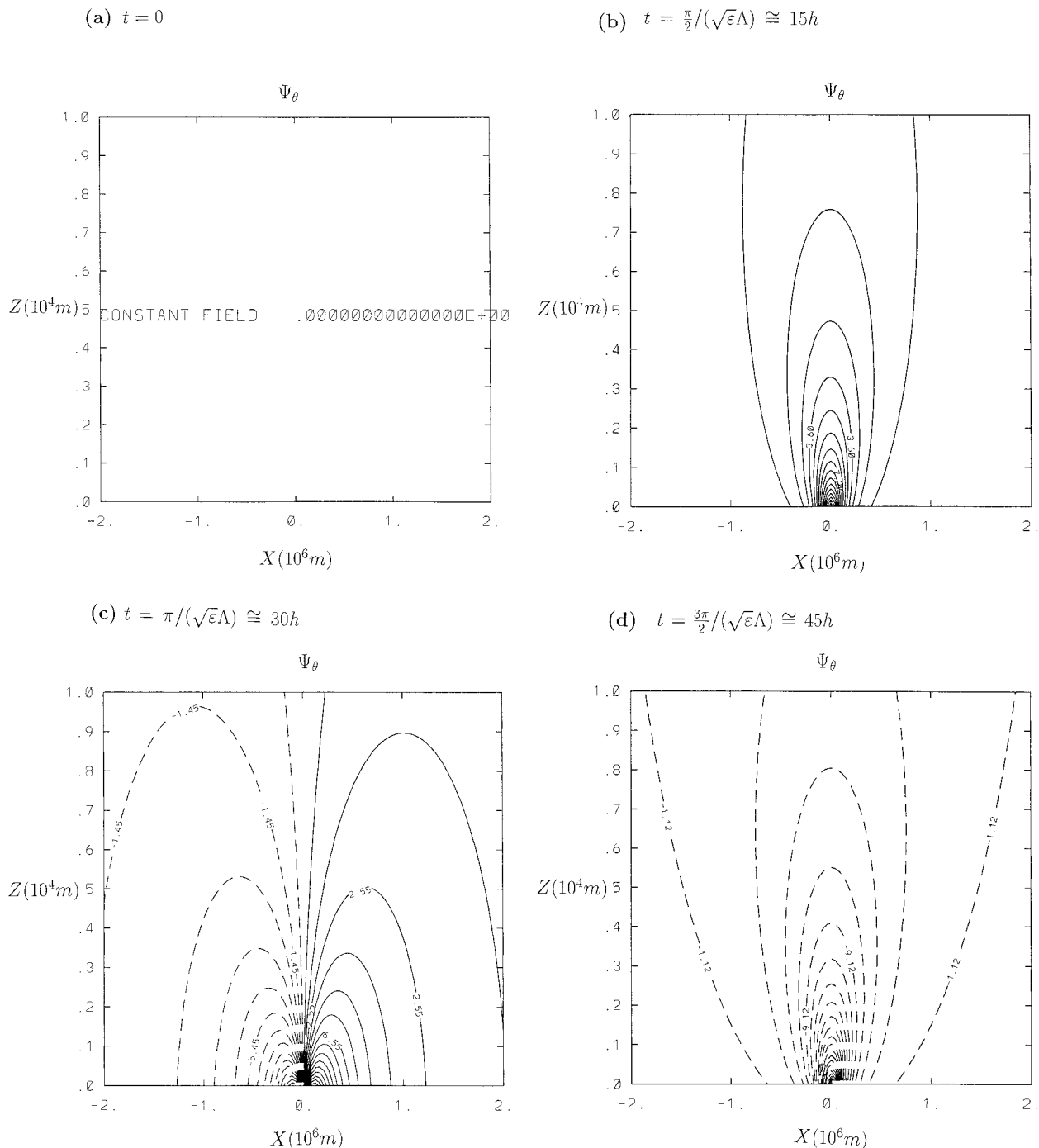
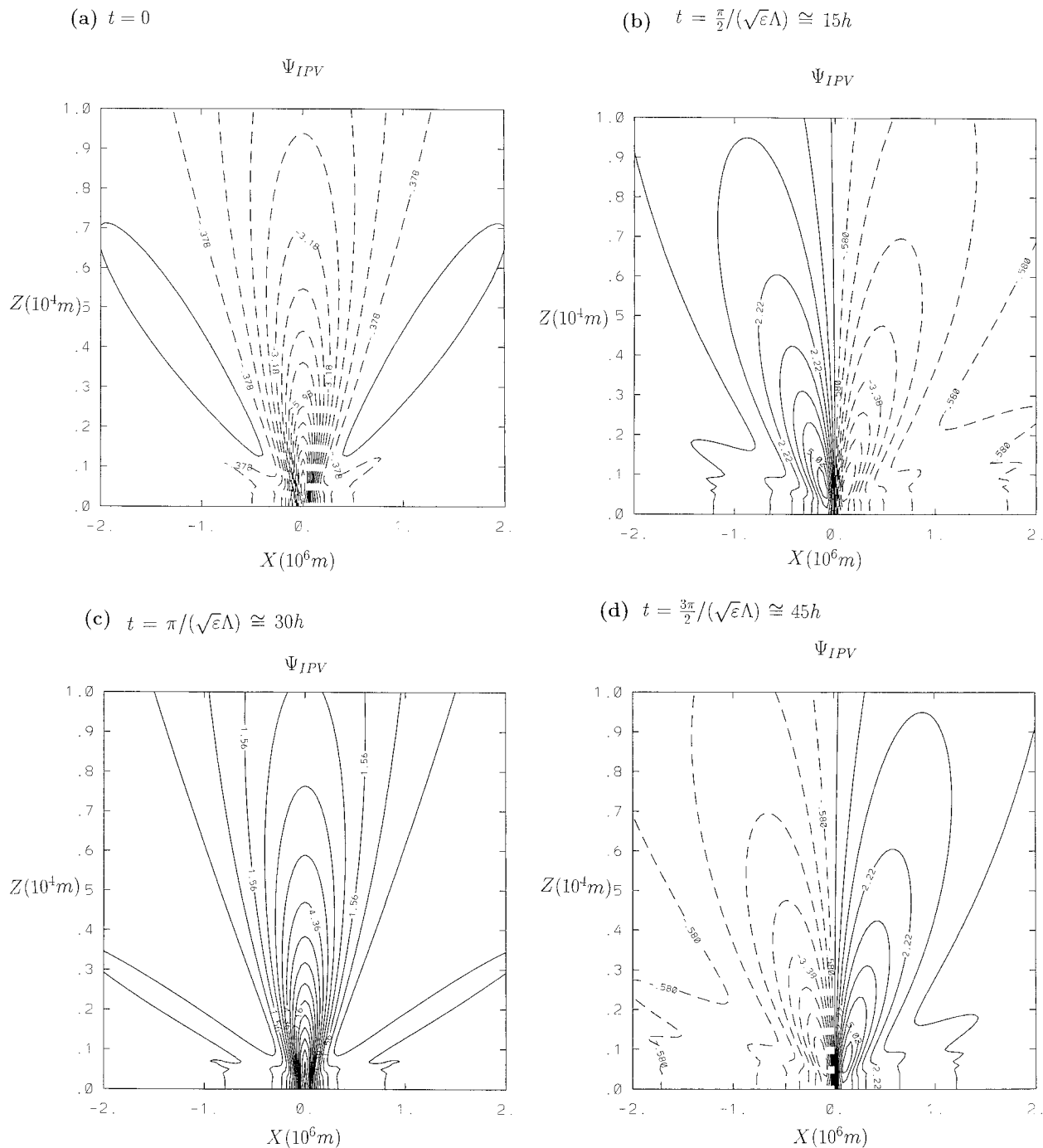


FIG. 4. As in Fig. 3, but for ψ_θ , Eq. (16).

$$\omega = \frac{k}{\mu} = \pm 1 \quad (\text{depending on the sign of } k). \quad (27)$$

The fact that the frequency and phase speed relations for waves in the semi-infinite Eady model and the regular Eady model become equivalent to more than two significant figures for waves with $\mu > 3.2 \times 10^{-6}$ ($\mu H > 3.2$) suggests that the influence of the tropopause is

negligible for these waves. Thus, the geophysically relevant wave solutions for the semi-infinite Eady model are the waves whose wavenumbers lie within the range $3.2 \times 10^{-6} < k < 2 \times 10^{-5}$. The corresponding dimensional wavelength range is $100\pi \text{ km} < \lambda < (2\pi/3.2) \times 1000 \text{ km}$. Excluding wavenumbers outside this range from the integral (24) isolates the geophysically relevant part of (24). For this range of wavenumbers, it is the

FIG. 5. As in Fig. 3, but for ψ_{IPV} , Eq. (15).

PV field between the dimensional heights $f_0/(2 \times 10^{-5} N) = 0.5$ km and $f_0/(3.2 \times 10^{-6} N) = 3.125$ km that is capable of causing resonant growth in surface wave packets that have zero group velocity.

It is also important to identify relevant time scales for our analysis. The closest atmospheric counterparts to our baroclinic zonal flow are the elongated baroclinic jets that often lie in storm track regions. It is not un-

common to observe large-scale jet streams remaining in approximately the same position for about a month. Could a natural selection process occur over such a period? To answer this question, we shall consider the nondimensional time period from $t = 0$ to $t = 100$ that corresponds to a 38.58-day period for the parameter values given in section 2a. Assuming that $\theta_0(k) = 0$, the initial surface streamfunction magnitude at the

wavenumber k would be given from (19), (6b), and (22) by

$$\varphi_{\text{IPV}}(\eta, k, x, 0, 0) = \mathcal{H}\left(\eta + \frac{1}{\mu}\right) \left[\frac{e^{-(1+\eta\mu)P}}{\mu} \right]. \quad (28)$$

Comparing (28) with (24) shows that the growth of the surface streamfunction field from $t = 0$ to $t = 100$ is proportional to the factor $\sin(100k\eta/2)/(\mu\eta/2)$. Thus wave packets that grow by more than 1 order of magnitude have η magnitudes less than the magnitude of η required to make

$$\frac{\sin\left(100\frac{k_{\text{max}}\eta}{2}\right)}{\frac{\mu_{\text{max}}\eta}{2}} = \frac{\sin(10^{-3}\eta)}{10^{-5}\eta} = 10. \quad (29)$$

Solving this equation for η reveals that it is only those wave packets with η magnitudes less than 2852.4 that can grow by more than 1 order of magnitude over the 38.58 day period; $\eta = 2852.4$ corresponds to a dimensionally correct group velocity of only 0.085 m s^{-1} or 285 km in 38.58 days. Evidently, the only PV wave packets capable of making surface streamfunction grow by an order of magnitude or more are those with a streamfunction group velocity that is close to zero.² Consequently, if the initial amplitude of streamfunction wave packets were evenly distributed across η values, absolutely unstable wave packets with approximately zero group velocity would emerge to dominate the streamfunction field by more than 1 order of magnitude.

Note that this naturally selected absolutely unstable part allows for any horizontal structure representable in terms of the geophysically relevant wavenumbers. (Since the cutting of a Fourier integral of wave functions at a specific wave number typically leads to a disturbance that is modulated by the wavenumber that it is cut off at, one might anticipate that these structures would be dominated by a wavelength close to the upper range of wavelengths over which the resonance can occur, i.e., around 2000 km.) The oscillation frequency of these selected wave packets is $(f/N)\Lambda$. Also note that the surface streamfunction amplitude of waves for which $\sin(10^{-5}\eta)$ is approximately equal to $10^{-5}\eta$ double in the first 9.26 h of development. In contrast, the most rapidly growing Eady mode in this basic state ($\lambda = 3927 \text{ km}$) takes about 24 h to double its surface amplitude. Thus, initial growth rates associated with PV resonance exceed those produced by the normal-mode mechanism.

We conclude then that absolute instability via PV res-

onance can occur in geophysically relevant flows and that it is not just a peculiarity of the particular solution described in section 2. In contrast to flows whose naturally selected structures are identifiable by their spatial structure and scale (such as normal modes), the naturally selected structures of the semi-infinite Eady basic state are identifiable by their group velocity and oscillation frequency. The horizontal scale of the selected structures depends on the details of the initial state. In order to illustrate that the above process is not solely confined to Eady's semi-infinite basic state, in appendix F we show how Lindzen's (1994) basic state of mean zero PV gradient with $\beta \neq 0$ also supports absolute instability.

4. Concluding remarks

The upstream expansion of a localized region of baroclinic eddy activity discussed in this paper is not seeded from within the wave packet but from PV disturbances far upstream of the localized wave packet. However, since these upstream PV disturbances have vanishingly small widths and make a vanishingly small impact on the streamfunction field, practical measurement error would render them undetectable in an atmospheric or laboratory environment. Practicable observations of an absolutely unstable wave packet of the type presented in this paper would not detect the upstream PV disturbances until they were in the localized streamfunction packet itself. Thus, observations would (erroneously) indicate that the PV disturbance appeared within the wave packet itself, that is, that the upstream expansion of the wave packet was seeded from within the streamfunction wave packet. In this way, our results show how easy it would be to mistake an absolute instability seeded from PV perturbations within the localized streamfunction wave packet for an instability that is seeded by upstream PV perturbations.

In Hoskins et al. (1985), it is noted that in quasigeostrophic theory, streamfunction anomalies fields look rather like smoothed PV anomalies. Superficially, one might suppose that this implies that streamfunction anomalies ought to move in the same direction as the PV anomalies that induce them. Here, we have shown that this is not the case. The movement of streamfunction anomalies depends on the group velocity of the wave packet defining the streamfunction anomaly. By placing infinitely thin PV waves of different wavelengths at different heights in the semi-infinite Eady model, one can define wave packets with a wide range of group velocities. This includes streamfunction wave packets whose group velocity is *westward* even though all of the PV responsible for inducing the wave packets is being advected *eastward*.

Discrete numerical models would be incapable of simulating the sustained upstream expansion and growth captured by our analytical model as they would be incapable of resolving the small vertical scales of the up-

² This result is consistent with Chang's (1992) analysis of the initial "near resonant" linear growth that results from a PV slab in the interior.

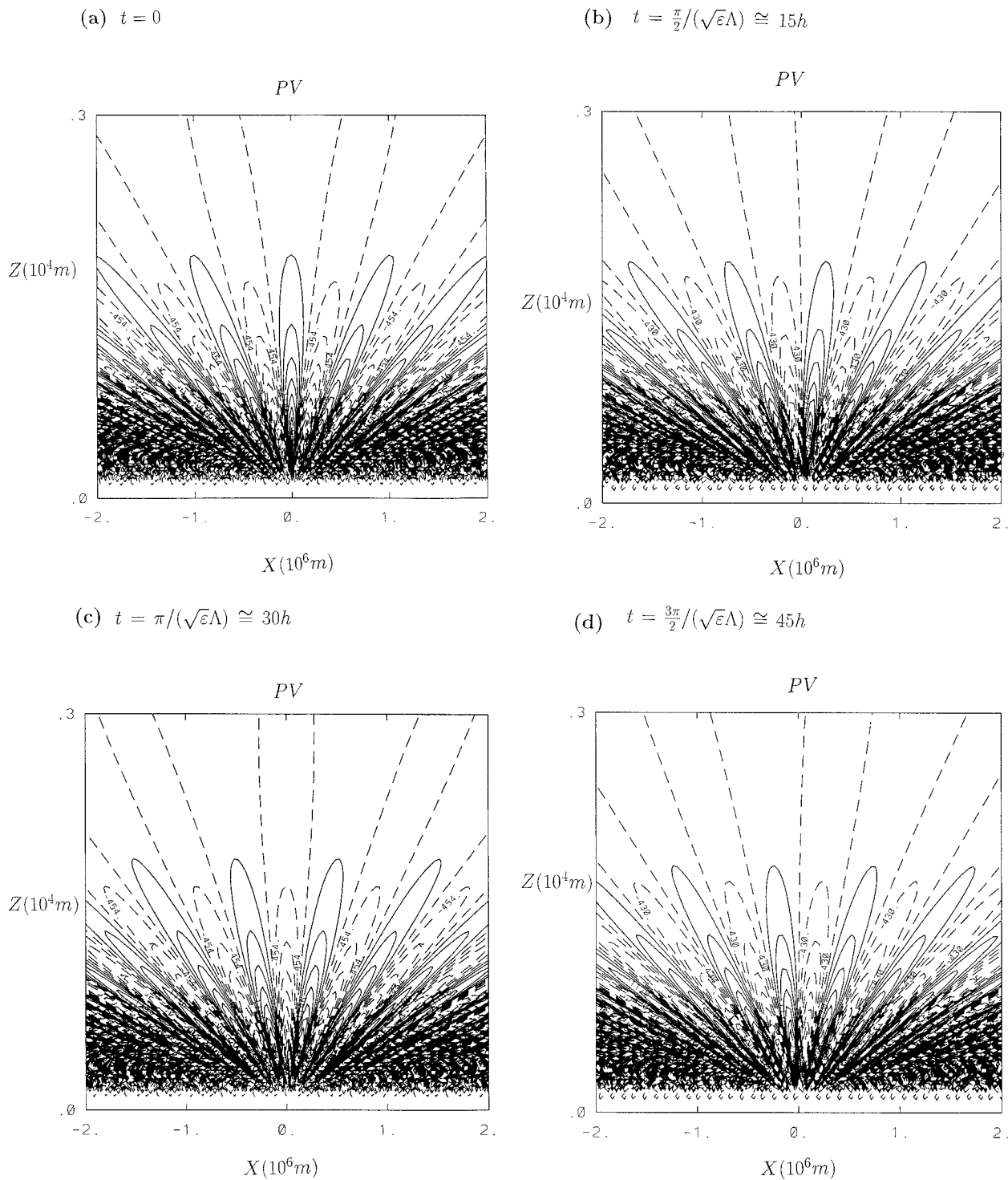


FIG. 6. As in Fig. 3, but for the PV field evolution, Eq. (17). The units of the PV field are normalized here by $A \times 10^{-18}$.

stream PV perturbations. They would, however, be capable of simulating a transient period of expansion and growth. Eventually, however, the upstream source of PV perturbations would be depleted and localized wave packet growth would cease. The inability of discrete

observational networks and discrete numerical models to capture the key upstream perturbations means that upstream perturbations of the type described in this study could contribute to weather forecasting errors.

The growth mechanism discussed in this paper relies

on the existence of finely layered PV structures. There are, however, a number of reasons why such finely layered PV disturbances would not be common in the atmosphere. First, diffusion of heat and momentum on the molecular scale would place a strict lower bound on the scale of the upstream PV disturbances. Second, eddy induced diffusion of heat and momentum would place a larger, though less well defined, lower bound on the scale of the upstream disturbances. Third, the mixing effects of vigorous large-scale midlatitude eddies would also decrease the lifetime of finely layered PV disturbances. Fourth, midlatitude flow is convoluted and zonally periodic; hence, the infinitely long fetch distance found in the semi-infinite Eady model does not exist in the atmosphere. On the other hand, moist convection, surface friction and radiative effects are constantly feeding anomalously high and low values of PV into the atmosphere. It is unlikely that the PV anomalies produced by these structures would never configure themselves in a way that would assist the growth of a baroclinic wave packet.

Shepherd (1985) examined the growth of disturbances to an infinite linear shear flow known as Couette flow. Despite superficial similarities, the stability of this flow is very different to that of the semi-infinite Eady model because it does not support edge waves let alone the resonant amplification of edge waves.

While the resonant interaction between interior PV and surface temperature waves is not limited to strictly two-dimensional disturbances (Davies and Bishop 1994), the group velocity of resonating waves with meridional variations is not independent of the zonal wavenumber nor is it equal to the speed of the wind at the ground. Thus, the wave packet behaviour discussed in this paper is only relevant to that subset of three-dimensional waves for which $k \gg l$. Furthermore, it is only relevant to situations in which one may neglect the horizontal component of the basic-state PV gradient. Lindzen (1994) has argued that there are many important situations for which this is a reasonable assumption.

Note that it is a trivial matter to turn our semi-infinite Eady model upside down and let the solid boundary represent the tropopause. Doing so leads to a system in which random PV perturbations amplify tropopause based wave packets that move at the speed of the wind at the tropopause. The observational work in Blackmon et al. (1984a,b), Lim and Wallace (1991), Lee and Held (1993), and Chang (1993) has demonstrated the existence of wave packets that show a resemblance to the wave packets that would be selected by such an inverted version of the semi-infinite Eady model. Specifically, they find wave packets with group velocities approximately equal to the speed of the wind at the tropopause. The semi-infinite Eady model or the Eady model itself does not, however, readily explain why the waves in these wave packets have wavelengths of around 3000–4000 km. The linear Eady model would predict that such waves would grow exponentially and have a group ve-

locity roughly equal to the speed of the midlevel wind—not the speed of the wind at the tropopause.

Our simple model does, however, suggest a mechanism for the origin of the medium-scale waves identified in observational data by Sato et al. (1993), Hirota et al. (1995), and Yamamouri et al. (1997). These waves typically have wavelengths of around 2000 km. Yamamouri et al. (1997) found that these waves occur at both upper levels and lower levels. The upper-level waves have phase speeds faster than the phase speeds of the long-wavelength waves. This characteristic mirrors that of the upper-level Eady neutral modes, which have phase speeds greater than the longer-wavelength troposphere filling modes. We have demonstrated that random PV perturbations will result in the growth of such waves. Thus, our theory is consistent with these observations. We also speculate that the lower-level medium-scale waves identified by Yamamouri et al. (1997) might be counterparts of the low-level Eady neutral modes which, as we have shown here, can also be excited by random PV perturbations.

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APPENDIX A

Algebraic Details Leading to Eq. (15)

From (6b),

$$\gamma\left(\frac{1}{\mu}, z\right) = \begin{cases} \frac{(e^{-\mu z} + e^{\mu z})}{2e} & \text{for } \frac{1}{z} > \mu \\ \frac{\beta e^{-\mu z}}{e} & \text{for } \frac{1}{z} < \mu, \end{cases} \quad (\text{A1})$$

where $\beta = (1 + e^2)/2$. Using (A1) and (14) in (13) gives

$$\begin{aligned} \psi_{\text{IPV}}(x, z, t) = & \cos(t) \int_{1/z}^{\infty} A e^{-\alpha_1 k} \beta \cos(kx) dk \\ & + \sin(t) \int_{1/z}^{\infty} A e^{-\alpha_1 k} \beta \sin(kx) dk \\ & + \cos(t) \int_0^{1/z} A \frac{(e^{-\alpha_1 k} + e^{-\alpha_2 k})}{2} \cos(kx) dk \\ & + \sin(t) \int_0^{1/z} A \frac{(e^{-\alpha_1 k} + e^{-\alpha_2 k})}{2} \sin(kx) dk, \end{aligned} \quad (\text{A2})$$

where $\alpha_1 = b + z$ and $\alpha_2 = b - z$. To perform the integrals indicated in (A2) we use the standard integral formulas

$$F_s(a, k) = \int e^{-ak} \sin(kx) dk = -\frac{e^{-ak}[a \sin(kx) + x \cos(kx)]}{a^2 + x^2}, \tag{A3}$$

$$F_c(a, k) = \int e^{-ak} \cos(kx) dk = \frac{e^{-ak}[x \sin(kx) - a \cos(kx)]}{a^2 + x^2}, \tag{A4}$$

where a is a constant. Using (A3) and (A4) in (A2) gives Eq. (15).

APPENDIX B

Limit Analysis of Singularities in Eq. (15)

The terms in Eq. (15)

$$\frac{\alpha_2}{\alpha_2^2 + x^2} + \frac{e^{-(\alpha_2/z)}}{\alpha_2^2 + x^2} \left[x \sin\left(\frac{x}{z}\right) - \alpha_2 \cos\left(\frac{x}{z}\right) \right] \text{ and } \tag{B1}$$

$$\frac{x}{\alpha_2^2 + x^2} - \frac{e^{-(\alpha_2/z)}}{\alpha_2^2 + x^2} \left[\alpha_2 \sin\left(\frac{x}{z}\right) + x \cos\left(\frac{x}{z}\right) \right] \tag{B2}$$

are both singular at $\alpha_2 = x = 0$. The complete set of lines that pass through this singular point may be written in the form

$$z = \beta x + b, \tag{B3}$$

where β gives the gradient of the line. By using (B3) in (B1) and (B2) and taking the limit as x tends to zero, we can determine the limit of (B2) and (B3) as one approaches the singularity along any line in the $x-z$ plane. Along the lines described by (B3), (B1), and (B2) take the respective forms

$$\frac{\beta x}{x^2(\beta + 1)} + \frac{e^{-[\beta x/(\beta x + b)]}}{x^2(\beta + 1)} \times \left[x \sin\left(\frac{x}{\beta x + b}\right) - \beta x \cos\left(\frac{x}{\beta x + b}\right) \right] \text{ and } \tag{B4}$$

$$\frac{x}{x^2(\beta + 1)} + \frac{e^{-[\beta x/(\beta x + b)]}}{x^2(\beta + 1)} \times \left[\beta x \sin\left(\frac{x}{\beta x + b}\right) + x \cos\left(\frac{x}{\beta x + b}\right) \right]. \tag{B5}$$

Taking the limit of (B4) and (B5) as x tends to zero gives

$$\lim_{x \rightarrow 0} \frac{\beta x}{x^2(\beta^2 + 1)} + \frac{\left(1 - \frac{\beta x}{b}\right)}{x^2(\beta^2 + 1)} \left(\frac{x^2}{b} - \beta x\right) = \frac{1}{b} \text{ and } \tag{B6}$$

$$\lim_{x \rightarrow 0} \frac{x}{x^2(\beta + 1)} - \frac{\left(1 - \frac{\beta x}{b}\right)}{x^2(\beta + 1)} \left(\frac{\beta x^2}{b} + x\right) = 0. \tag{B7}$$

In this way, we see that ψ_{IPV} is bounded at $\alpha_2 = x = 0$. Equation (15) also contains terms that are singular at $z = 0$. Near $z = 0$, all of these terms take the general form

$$e^{-b/z} \left[x \sin\left(\frac{x}{z}\right) - b \cos\left(\frac{x}{z}\right) \right]. \tag{B8}$$

Since $e^{-b/z}$ approaches zero faster than any other polynomial function of z as z approaches zero, ψ_{IPV} and all of its derivatives are continuous. Since a term very similar to the above term appears in the PV equation (17), one may similarly deduce that the PV and all its derivatives are also continuous.

APPENDIX C

Derivation of Eq. (17)

Using (11) in (6a) gives

$$PV(x, z, t) = \int_{-\infty}^{\infty} \int_0^{\infty} G(x, z, t; z', k) \delta\left(z' - \frac{1}{\mu}\right) dz' dk, \tag{C1}$$

where

$$G(x, z, t; z', k) = F(z') \delta(z' - z) e^{ik(x-z't)}. \tag{C2}$$

Integrating (C1) gives

$$PV(x, z, t) = \int_{-\infty}^{\infty} F\left(\frac{1}{\mu}\right) \delta\left(\frac{1}{\mu} - z\right) \left[\cos(t) - i \frac{k}{\mu} \sin(t) \right] \times [\cos(kx) + i \sin(kx)] dk. \tag{C3}$$

Since $F(1/\mu)$ is an even function of k , we may rewrite this integral in terms of just two integrals from 0 to ∞ . In addition, we can change the variable of integration from k to $1/\mu$. Doing so leads to the formulas

$$PV(x, z, t) = 2 \cos(t) \int_0^{\infty} \mu^2 F\left(\frac{1}{\mu}\right) \delta\left(\frac{1}{\mu} - z\right) \cos(\mu x) d\left(\frac{1}{\mu}\right) + 2 \sin(t) \int_0^{\infty} \mu^2 F\left(\frac{1}{\mu}\right) \delta\left(\frac{1}{\mu} - z\right) \sin(\mu x) d\left(\frac{1}{\mu}\right) = 2 \left[\cos(t) \frac{F(z)}{z^2} \cos\left(\frac{x}{z}\right) + \sin(t) \frac{F(z)}{z^2} \sin\left(\frac{x}{z}\right) \right]. \tag{C4}$$

From (14),

$$F(z) = -\frac{Ae^{[1-(b/z)]}}{2z}. \tag{C5}$$

Using (C5) in (C4) yields (17).

APPENDIX D

Changing Variables from (z', k) to (η, k)

Noting that the Jacobian of the coordinate transformation $\partial(z', k)/\partial(\eta, k) = 1$, (6b) may be rewritten in the form

$$\begin{aligned} \psi_{\text{IPV}}(x, z, t) &= \int_{k=-\infty}^{\infty} \int_{\eta=-1/\mu}^{\infty} -\frac{\gamma[z'(\eta, k), z]}{\mu} P[z'(\eta, k), k] \\ &\quad \times e^{-i(k/\mu)t} e^{ik(x-\eta t)} d\eta dk, \end{aligned} \quad (\text{D1})$$

or equivalently

$$\begin{aligned} \psi_{\text{IPV}}(x, z, t) &= \int_{k=-\infty}^{\infty} \int_{\eta=-\infty}^{\infty} -\mathcal{H}\left(\eta + \frac{1}{\mu}\right) \frac{\gamma[z'(\eta, k), z]}{\mu} \\ &\quad \times P[z'(\eta, k), k] e^{-i(k/\mu)t} e^{ik(x-\eta t)} d\eta dk, \end{aligned} \quad (\text{D2})$$

where \mathcal{H} is the unit step function. Changing the order of integration in this equation gives Eq. (22).

$$\psi_{\theta}(x, z, t) = \int_{\xi=0}^{\infty} \left\{ \int_{k=-\infty}^{\infty} \frac{iP\left(\frac{\xi}{\mu}, \mu\right) e^{-\mu z'}}{\frac{\mu^2}{2}(1-\xi)} \sin\left[\frac{k(1-\xi)}{2\mu} t\right] e^{-i[k(1+\xi)t/2\mu]} e^{-\mu z} e^{ikx} d\mu \right\} d\xi. \quad (\text{E3})$$

In this case, equations describing the resonant amplification of surface modes are obtained by letting ξ tend to 1 and follow a similar procedure to that given in section 3b.

APPENDIX F

A Mean Zero PV Gradient, but with $\beta \neq 0$

For a Boussinesq baroclinic quasigeostrophic basic state on a β plane with constant stratification, the meridional mean PV gradient is

$$\frac{\partial \bar{P}}{\partial y} = \beta - \bar{U}_{zz}. \quad (\text{F1})$$

Thus, in order to keep a zero mean PV gradient, we

APPENDIX E

Partitioning into Coherent Wave Packets with $c_g = 0$

The coordinate transformation

$$z'(\xi, k) = \frac{\xi}{\mu}, \quad k(\xi, k) = k, \quad \frac{\partial(z', k)}{\partial(\xi, \mu)} = \frac{1}{\mu} \quad (\text{E1})$$

allows (6b) to be rewritten in the form

$$\begin{aligned} \psi_{\text{IPV}}(x, z, t) &= \int_{\xi=0}^{\infty} \left\{ \int_{k=-\infty}^{\infty} -\frac{\gamma\left(\frac{\xi}{\mu}, z\right)}{\mu^2} P\left(\frac{\xi}{\mu}, \mu\right) e^{i[kx - (k\xi/\mu)t]} dk \right\} d\xi. \end{aligned} \quad (\text{E2})$$

The integral inside the large curly brackets describes a wave packet, with zero group velocity, which oscillates with a dimensional period, $T = 2\pi N/f\Lambda\xi$. Thus Eq. (E2) gives a representation of the IPV part as a sum of wave packets, each of which has zero group velocity but a different period of oscillations. Making a similar change of variables in (7a) allows us to see how the response of the surface temperature field depends on the time period of the IPV wave packet that induces the response. The amplitude of this response is entirely described by the variable B . Using (E1) in (9b) to obtain the expression for B in terms of k , ξ , and t yields

should set $\bar{U}_{zz} = \beta$. This can be done, following Lindzen (1994), by letting $\partial\bar{U}/\partial z$ be a function of z . Assuming that $\bar{U}(z=0) = 0$ yields a mean wind profile of the form

$$\bar{U}(z) = m_0 z \left(1 + \frac{\beta z}{2m_0} \right), \quad (\text{F2})$$

where $m_0 = (\partial\bar{U}/\partial z)(z=0)$.

Since in this basic state, perturbations are governed by (1), (2), and (3), the discrete-spectrum surface temperature wave takes the same form as that given by Eqs. (7a,b). The equation for the evolution of the interior PV and its associated streamfunction field is obtained by replacing $\Lambda z'$ by the \bar{U} field described by (F2) in Eqs. (6a) and (6b), respectively. For the case where the initial

surface temperature perturbation is zero, substituting (6b) and (7a) into (3b) and solving for B yields

$$B(k, z', t) = \frac{-i\gamma(z', 0)P(z', k)}{\frac{k}{2}\left[\mu^{-1} - z'\left(1 + \frac{\beta z'}{2m_0}\right)\right]} \times \sin\left\{\frac{\mu}{2}\left[\mu^{-1} - z'\left(1 + \frac{\beta z'}{2m_0}\right)\right]t\right\} \times \exp\left\{-i\frac{k}{2}\left[\mu^{-1} + z'\left(1 + \frac{\beta z'}{2m_0}\right)\right]t\right\}. \quad (\text{F3})$$

From inspection of (F3) one can see that the height resonant growth is obtained when

$$\frac{\beta z'^2}{2m_0} = \mu^{-1} - z'. \quad (\text{F4})$$

Since the left-hand side of this equation is positive definite, resonant growth is obtained when z' is less than $1/\mu$, that is, less than the resonant height for the $\beta = 0$ basic state. This could have been anticipated from the fact that the mean wind of the parabolic profile is, at all heights, larger than the mean wind of the linear profile. Consequently, the height at which the mean wind matches the free surface temperature wave must be lower than $1/\mu$. Apart from this change in the height at which PV waves must be placed in order to induce linear growth, all of the arguments applied to the basic state with linear shear can also be applied to the $\beta \neq 0$ basic state. Thus, arbitrary disturbances to the $\beta \neq 0$ basic state would also produce absolutely unstable streamfunction wave packets.

REFERENCES

- Bishop, C. H., and A. J. Thorpe, 1994: Potential vorticity and the electrostatics analogy: Quasi-geostrophic theory. *Quart. J. Roy. Meteor. Soc.*, **120**, 713–731.
- Blackmon, M. L., 1976: A climatological spectral study of the 500 mb geopotential height of the Northern Hemisphere. *J. Atmos. Sci.*, **33**, 1607–1623.
- , Y.-H. Lee, and J. M. Wallace, 1984a: Horizontal structure of 500 mb height fluctuations with long, intermediate and short time scales. *J. Atmos. Sci.*, **41**, 961–979.
- , —, —, and H.-H. Hsu, 1984b: Time variation of 500 mb height fluctuations with long, intermediate and short time scales as deduced from lag-correlation statistics. *J. Atmos. Sci.*, **41**, 981–991.
- Briggs, R. J., 1964: *Electron Stream Interaction with Plasma. Research Monogr.*, No. 29, The MIT Press, 187 pp.
- Chang, E. K. M., 1992: Resonating neutral modes of the Eady model. *J. Atmos. Sci.*, **49**, 2452–2463.
- , 1993: Downstream development of baroclinic waves as inferred from regression analysis. *J. Atmos. Sci.*, **50**, 2038–2053.
- Davies, H. C., and C. H. Bishop, 1994: Eady edge waves and rapid development. *J. Atmos. Sci.*, **51**, 1930–1946.
- Eady, E. T., 1949: Long waves and cyclone waves. *Tellus*, **1**, 33–52.
- Farrell, B. F., 1984: Modal and non-modal baroclinic waves. *J. Atmos. Sci.*, **41**, 668–673.
- Gill, A. E., 1982: *Atmosphere–Ocean Dynamics*. Academic Press, 662 pp.
- Hirota, I., K. Yamada, and K. Sato, 1995: Medium-scale travelling waves over the North Atlantic. *J. Meteor. Soc. Japan*, **73**, 1175–1179.
- Holton, J. R., 1992: *An Introduction to Dynamic Meteorology*. Academic Press, 507 pp.
- Hoskins, B. J., and P. J. Valdes, 1990: On the existence of stormtracks. *J. Atmos. Sci.*, **47**, 1854–1864.
- , M. E. McIntyre, and W. Robinson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.*, **111**, 877–946.
- James, I. N., 1994: *Introduction to Circulating Atmospheres*. Cambridge University Press, 422 pp.
- Lau, N. C., and J. M. Wallace, 1979: On the distribution of horizontal transports by transient eddies in the Northern Hemisphere wintertime circulation. *J. Atmos. Sci.*, **36**, 1844–1863.
- Lee, S., and I. M. Held, 1993: Baroclinic wave packets in models and observations. *J. Atmos. Sci.*, **50**, 1413–1428.
- Lim, G. H., and J. M. Wallace, 1991: Structure and evolution of baroclinic waves as inferred from regression analysis. *J. Atmos. Sci.*, **48**, 1718–1732.
- Lindzen, R. S., 1994: The Eady problem for a basic state with zero PV gradient but $\beta \neq 0$. *J. Atmos. Sci.*, **51**, 3221–3226.
- Merkine, L., 1977: Convective and absolute instability of baroclinic eddies. *Geophys. Astrophys. Fluid Dyn.*, **9**, 129–157.
- Müller, J. C., H. C. Davies, and C. Schar, 1989: An unsung mechanism for frontogenesis and cyclogenesis. *J. Atmos. Sci.*, **46**, 3664–3672.
- Parker, D. J., 1998: Secondary frontal waves in the North Atlantic region: A dynamical perspective of current ideas. *Quart. J. Roy. Meteor. Soc.*, **124**, 829–856.
- Pedlosky, J., 1964: An initial value problem in the theory of baroclinic instability. *Tellus*, **16**, 12–17.
- Pierrehumbert, R. T., 1984: Local and global baroclinic instability of zonally varying flow. *J. Atmos. Sci.*, **41**, 2141–2162.
- Sato, K., H. Eito, and I. Hirota, 1993: Medium-scale travelling waves over the North Atlantic. *J. Meteor. Soc. Japan*, **71**, 427–436.
- Shepherd, T. G., 1985: Time development of small disturbances to plane Couette flow. *J. Atmos. Sci.*, **42**, 1868–1871.
- Swanson, K. L., and R. T. Pierrehumbert, 1995: Potential vorticity homogenization and stationary waves. *J. Atmos. Sci.*, **52**, 990–994.
- Thorncroft, C. D., and B. J. Hoskins, 1990: Frontal cyclogenesis. *J. Atmos. Sci.*, **47**, 2317–2336.
- Yamamori, M., K. Sato, and I. Hirota, 1997: A study of the seasonal variation of upper tropospheric medium-scale waves over east Asia based on regional climate model data. *J. Meteor. Soc. Japan*, **75**, 13–22.